A note on the statistics of stellar quadrangles

Recall that we are making quadrangles ABCD of stars, where A and B are the two most widely separated, and C and D are two more stars, required to fall within the square box of which A and B define the diagonally separated corners.

If the angular separation of A and B is θ_{AB}, then in the small-angle approximation, the square in which C and D must fall has solid angle

$$\Omega_{\mathsf{AB}} = \frac{1}{2}\,\theta_{\mathsf{AB}}^2 = Q\,\theta_{\mathsf{AB}}^2 \quad , \tag{1}$$

where the factor 1/2 has been symbolized "Q" in case we switch to a different-shaped restriction on C and D. If the density of stars on the sky (number per solid angle) is Γ, and we assume that *there is no angular clustering of stars*, then the mean number of stars ν in this angular patch is $\nu = \Gamma\,\Omega_{\mathsf{AB}}$, and the actual number n of stars in the patch is (again assuming no clustering) drawn from the Poisson distribution, to wit

$$p(n) = \frac{\nu^n\,e^\nu}{n!} \quad . \tag{2}$$

We can only make a quad if $n \geq 2$, in which case in fact we have $\binom{n}{2}$ unique choices for C and D.

Once A and B have been chosen, the expectation value $E(N_{\mathsf{CD}})$ for the number of unique quads (choices of C and D) is found by summing over all values of n, or

$$\begin{aligned} E(N_{\mathsf{CD}}) &= \sum_{n=2}^{\infty} \binom{n}{2} \frac{\nu^n\,e^\nu}{n!} && (3)\\ &= \frac{1}{2}\,\nu^2 \sum_{n=0}^{\infty} \frac{\nu^n\,e^\nu}{n!} && (4)\\ &= \frac{1}{2}\,\nu^2 && (5)\\ &= \frac{1}{2}\,Q^2\,\Gamma^2\,\theta_{\mathsf{AB}}^4 \quad , && (6) \end{aligned}$$

where the sum from 2 to ∞ is converted to a sum from 0 to ∞ by dividing out a ν^2 and some combinatoric factors. I (DWH) feel certain that we could have guessed this result up-front with some clever argument.

We now know how many CD pairs there are, given an AB pair. How many AB pairs are there? For each choice of A (*again* assuming no clustering), the expectation value $E(N_{\mathsf{B}})$ of the number of choices of B in an annulus of radius θ_{AB} and width $\mathrm{d}\theta$ is (again in the small-angle approximation)

$$E(N_{\mathsf{B}}) = 2\pi\,\Gamma\,\theta_{\mathsf{AB}}\,\mathrm{d}\theta \quad . \tag{7}$$

The expectation value $E(N_{\mathsf{BCD}})$ of the number of quads that can be made with this choice of A at this angular separation is then

$$E(N_{\mathsf{BCD}}) = \pi\,Q^2\,\Gamma^3\,\theta_{\mathsf{AB}}^5\,\mathrm{d}\theta \quad . \tag{8}$$

The total number of choices for A is the density Γ times the total survey solid angle $\Omega_{\rm tot}$; the total number of unique quads is the product of this and *half* of $E(N_{\mathsf{BCD}})$ (because each AB choice can be swapped). So the expectation value $E(N_{\mathsf{ABCD}})$ for the total number of quads is

$$E(N_{\mathsf{ABCD}}) = \frac{\pi\,Q^2}{2}\,\Omega_{\rm tot}\,\Gamma^4\int_{\theta_{\rm min}}^{\theta_{\rm max}}\theta^5\,{\rm d}\theta \tag{9}$$

$$E(N_{\mathsf{ABCD}}) = \frac{\pi\,Q^2}{2}\,\frac{N_{\rm tot}^4}{\Omega_{\rm tot}^3}\int_{\theta_{\rm min}}^{\theta_{\rm max}}\theta^5\,{\rm d}\theta \quad , \tag{10}$$

where the first form is in terms of the angular number density Γ and the second is in terms of the total number of stars $N_{\rm tot}$, and where AB pairs are allowed to come from any angular separation from $\theta_{\rm min}$ to $\theta_{\rm max}$. In both of these expressions we have implicitly assumed that the angular density of stars is constant on the sky.