1 Covariant formulation of EM

1.1 Special relativity

At the beginning of the 1900s physicists were struggling with the incompatibility between:

- the relativity principle, which says that the laws of physics must be the same in all inertial frames (IF)
- Galilean transformation (GT), that at the time gave the rules going from one IF to another
- Maxwell’s equations, which describe electromagnetism

The incompatibility arises because Maxwell’s equations are not invariant under Galilean transformations, thus breaking the principle of relativity. There are many ways of seeing this.

- Maxwell’s equation feature explicitly the speed of light \(c\) as a parameter. But according to Galilean transformation, the speed of light should be different in different IF’s that move relative to each other which implies \(c\) is only speed of light in a “special frame of reference” in which Maxwell’s equation hold.

- The wave equation for EM potentials/fields is not invariant (i.e. it changes form) when one goes from on IF to another using Galilean transformation. Indeed, in Galilean transformations,

\[
\vec{r}' = \vec{r} - \vec{v} t \quad t' = t \quad \vec{v}: \text{the relative velocity between IF’s}
\]

while the wave equation for any scalar \(f\):

\[
\square f(\vec{r}, t) = \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) f(\vec{r}, t) = 0
\]

Change as we go to \(\vec{r}'\) coordinates as:

\[
\nabla_i = \frac{\partial}{\partial p_i} = \frac{\partial r'_j}{\partial r_i} \frac{\partial}{\partial r'_j} + \frac{\partial t'}{\partial t} \frac{\partial}{\partial t'} = \delta_{ij} \frac{\partial}{\partial r'_j} + 0 \frac{\partial}{\partial t'} = \frac{\partial}{\partial r'_i} = \nabla'_i
\]

\[
\frac{\partial}{\partial t} = \frac{\partial r'_i}{\partial t} \frac{\partial}{\partial r'_i} + \frac{\partial t'}{\partial t} \frac{\partial}{\partial t'} = -v_i \frac{\partial}{\partial r'_i} - 1 \cdot \frac{\partial}{\partial t'} = \frac{\partial}{\partial t'} - \vec{v} \cdot \nabla
\]

\[
\Rightarrow \left[ \nabla'^2 - \frac{1}{c^2} \left( \frac{\partial}{\partial t'} - \vec{v} \cdot \nabla' \right)^2 \right] f(\vec{r}', t') = 0
\]
or
\[
\left[ \square' + \frac{2}{c^2} \frac{\vec{v} \cdot \nabla}{c} \frac{\partial}{\partial t'} - \left( \frac{\vec{v} \cdot \nabla}{c} \right)^2 \right] f(\vec{r}', t') = 0
\] (6)

which clearly shows that the form of the wave equation is not preserved by Galilean transformations. Furthermore, if we take e.g. \( \vec{v} = v \hat{z} \) and consider waves propagating in the +\( \hat{z} \) direction, a solution to this equation is just, as expected from GT’s
\[
f(\vec{r}', t') = g(z' - ct' + vt') \text{ with } g \text{ arbitrary scalar function of single variable}
\] (7)

indeed:
\[
\begin{align*}
\nabla'^2 g &= g'' \\
-\frac{1}{c^2} \frac{\partial^2 g}{\partial t'^2} &= -\frac{1}{c^2} (v - c)^2 g'' \\
\frac{2}{c^2} v \frac{\partial}{\partial z'} \frac{\partial}{\partial t'} g &= \frac{v}{c^2} (v - c) v g'' \\
-\frac{1}{c^2} v^2 \frac{\partial^2 g}{\partial z'^2} &= -\frac{v^2}{c^2} g''
\end{align*}
\]
\[\Rightarrow [\cdots] g = \left[ 1 - \frac{(v - c)^2}{c^2} + \frac{2v}{c} - \frac{v^2}{c^2} \right] g'' = 0 \quad \checkmark
\] (8)

If \( v = c \), the solution becomes \( f(\vec{r}', t) = g(z') \), i.e. an observer in rest frame \( S' \) sees no wave propagation at all.

\[\bullet\] another way of seeing incompatibility between i)-iii) above is to consider a simple EM configuration from the point of view of different IF’s and arriving at inconsistencies when we use GT’s to go from one to the other. The system is made of two cylinders, one with equal positive and negative charges and the other with only positive charges. In both cylinders, positive charges (of the same size) move with the same velocity:

Since \( Q = 0 \) in the top cylinder, there is no electric force. However, in the IF whose \( \oplus \) charges move, there is an attractive force between the parallel currents in the cylinders. In the frame where \( \oplus \) charges are at rest, \( Q = 0 \) still, so no electric force but now the magnetic force has disappeared since there is no current in the lower cylinder. Thus, we arrive at a contradiction.

The leading explanation at that time for these inconsistencies was that the principle of relativity did not hold for EM, and that there was a special frame in which Maxwell’s equations hold (this special frame was at rest with respect to a mechanical medium, so-called “ether”, that made possible light propagation). In another IF, one should see the speed of light vary as dictated by GT’s as we saw in the wave example above. Therefore, by measuring the speed of EM waves one could find out the relative velocity of any system with respect to the special frame. However, all such experiments failed (the best known of which was the
Michelson-Morley experiment), returning always the same value for the speed of light independently of the motion of the observer. These results led Poincare (in 1899, 1900 and 1904) to suggest that absolute motion cannot be detected by experiments of any kind, including EM (thus implying that i) and iii) above are correct). In 1904, Lorentz discovered a curious and remarkable transformation of space and time variables that left the Maxwell’s equations invariant but the connection to replacing ii) (GT’s) by then was not made (in fact, he tried to make his transformation consistent with the picture of special I.F.).

In 1905, Einstein developed the special theory of relativity from two basic postulates: 1) the relativity principle and 2) the constancy of speed of light (independent of the motion of IF). In doing so, he derived Lorentz transformation (LT) from these 2 postulates, replaced GT’s by LT’s and made i), ii), iii) above consistent. These transformation led to completely different notions of how space and time are related to each other and to the concept of spacetime itself (Minkowski 1907). And because Newtonian notions of space and time are modified, the laws of mechanics needed to be changed as well.

Let us now see how Lorentz transformation (LT) follow from the relativity principle and constancy of the speed of light. Consider two IF’s, one $S$ with coordinates $x, y, z \& \text{ times } t$, and the other $S'$ moving relative to $S$ with relative velocity in the $x$ direction. Perpendicular direction are then equivalent and $y' = y$, $z' = z$ [otherwise one runs into inconsistencies, as it’s easy to show: e.g. consider two hollow cylinders moving in $\hat{x}$ in opposite directions in center of mass frame and of the same transverse size in that frame: they will collide. However if transverse lengths depend on IF, in rest frame of one of the cylinders one goes inside the other (no collision) which is inconsistent].

Consider a free particle in $S$, then we know that such particle has a linear relationship between $x \& t$, i.e.

$$x = x_0 + vt$$  \hspace{1cm} (10)

From this, it follows that the relationship between $x' \& t'$ must also be linear since the principle of relativity requires that all IF be equivalent. Therefore, the relation between primed and non-primed coordinates must be linear, otherwise free particle motion will look different in different IF’s. Then we have e.g.

$$x = \gamma(x' + \alpha t)$$  \hspace{1cm} (11)

Now, the equation of motion of $x = 0$ in $S'$ is,

$$0 = \gamma(x' + \alpha t') \implies x' = -\alpha t'$$  \hspace{1cm} (12)

but this must be $x' = -Vt'$, $\Rightarrow \alpha = V$, then

$$x = \gamma(x' + Vt')$$  \hspace{1cm} (13)

Now, by symmetry (again, due to the principle of relativity) we must have analogously for $x'$ as a function of $x, t$:

$$x' = \gamma(x - Vt)$$  \hspace{1cm} (14)
(i.e. just changing $V \to -V$) and by linearity $\gamma$ cannot depend on coordinates (otherwise the transformation becomes non-linear). It can only depend on the relative velocity $\vec{V}$, and by isotropy in fact only on its magnitude $V$, $\Rightarrow \gamma = \gamma(V)$

Now, we are ready to use the second postulate, constancy of the speed of light. Suppose that when the origins of $S$ & $S'$ coincide (let’s call this $t = t' = 0$ a light ray propagating in $+\hat{x}$ (+$\hat{x'}$) direction in launched. By constancy of speed of light we have $x = ct$, $x' = ct'$ then

$$\begin{cases} ct = \gamma(c + V)t' \Rightarrow ct = \gamma(c + V)\gamma(c - V)\frac{t}{c} \\ ct' = \gamma(c - V)t \Rightarrow c^2 = \gamma^2(c^2 - V^2) \end{cases}$$

$$\Rightarrow \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \beta^2}} \beta = \frac{v}{c} \tag{16}$$

Then we have:

$$x = \gamma(x' + Vt), \quad x' = \gamma(x - Vt) \tag{17}$$

and by combining these two we also obtain, after simple algebra

$$ct = \gamma(ct' + \beta x'), \quad ct' = \gamma(ct - \beta x) \tag{18}$$

These are the Lorentz transformations, derived only from the principle of relativity & constancy of the speed of light (i.e. independently of Maxwell’s equations). Note that when $c \ll c$, $\beta \ll 1$ and $\gamma \approx 1$

$$x \approx x' + Vt', \quad x' \approx x - Vt, \quad t' = t \tag{19}$$

If a particle is moving with velocity $v' = \frac{dx'}{dt}$ in $S'$ we have

$$\frac{v'}{c} = \frac{1}{\gamma} \frac{dx'}{dt} - \frac{V}{c} \frac{dt}{dt} = \frac{\nabla}{\gamma} - \frac{V}{c} \frac{\nabla}{\beta \frac{dx}{dt}} \Rightarrow v' = \frac{\nabla}{1 - \frac{v}{c}} \tag{20}$$

So, addition of velocities does not work anymore, as expected. In particular, if $v = c$, $v' = \frac{c - v}{1 - \frac{v}{c}} = c \sqrt{2}$ as it should due to consistency of speed of light. Note however that when $\frac{v}{c}, \frac{V}{c} \ll 1$ we recover $v' \approx v - V$, i.e. Galilean addition of velocities.

We can now go back to the wave equation and check that it is invariant under LT’s. Indeed.

$$\frac{\partial}{\partial x} \partial x' \partial + \frac{\partial'}{\partial x} \partial' = \gamma \frac{\partial}{\partial x'} - \gamma \frac{\partial}{\partial x'} \frac{\beta}{c} \frac{\partial}{\partial t'}$$

$$\frac{\partial}{\partial t} = \gamma \frac{\partial}{\partial x'} + \frac{\partial}{\partial t'} \frac{\beta}{c} \frac{\partial}{\partial x'} \frac{\partial}{\partial t'} = \gamma \frac{\partial}{\partial x'} - \gamma \frac{\partial}{\partial x'} \frac{\beta}{c} \frac{\partial}{\partial t'} \tag{21}$$

$$\frac{\partial^2}{\partial x^2} = \gamma^2 \frac{\partial^2}{\partial x'^2} - 2\gamma \frac{\partial}{\partial x'} \frac{\partial}{\partial x'} + \gamma \frac{\partial^2}{\partial t'^2} \tag{23}$$

$$- \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = - \gamma^2 \frac{\partial^2}{\partial x'^2} - \frac{\gamma^2}{c} \frac{\partial}{\partial t'} \frac{\partial}{\partial t'} - \gamma \frac{\partial^2}{\partial x'^2} \tag{24}$$

Since $\gamma^2(1 - \beta^2) = 1$ and $\frac{\partial^2}{\partial y'^2} = \frac{\partial^2}{\partial z'^2}$ and $\frac{\partial^2}{\partial z'^2} = \frac{\partial^2}{\partial z'^2}$, we indeed obtain

$$\square^2 = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = \square' = \nabla'^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t'^2} \Rightarrow \square = \text{inv} \tag{25}$$
A closely related invariant is the so-called interval \( ds^2 \):

\[
ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)
\]  \hspace{1cm} (26)

In our case \( dy = dy', \, dz = dz' \). so all we have to do is to check invariance of \( c^2 dt^2 - dx^2 \). We have:

\[
dx = \gamma(dx' + V \, dt') \\
dt = \gamma(c \, dt' + \beta \, dx')
\]  \hspace{1cm} (27)

\[
\Rightarrow c^2 dt^2 - dx^2 = \gamma^2 \left[ (c \, dt' + \beta \, dx')^2 - (dx' + V \, dt')^2 \right]
\]  \hspace{1cm} (28)

\[
= \gamma^2 \left[ c^2 dt'^2 + 2 \beta c \, dt' \, dx' + \beta^2 dx'^2 - dx'^2 - 2V \, dx' \, dt' - V^2 dt'^2 \right]
\]  \hspace{1cm} (29)

\[
= \gamma^2 \left[ c^2 dt'^2 \left( 1 - \beta^2 \right) - dx'^2 \left( 1 - \beta^2 \right) \right]
\]  \hspace{1cm} (30)

Lorentz transformation are a linear transformations connecting the space and time coordinates in one IF to the corresponding coordinates in another IF in uniform motion with respect to the first one. Therefore it is convenient to introduce spacetime coordinates at each inertial frame \( x^\mu = (ct, x, y, z) \), \( \mu = 0, 1, 2, 3 \) to characterize events (things that happen at a given time & space) and put time and spatial coordinates in an equal footing. A different observer in motion with respect to such inertial frame will have its own spacetime coordinates \( x'^\mu = (ct', x', y', z') \) “attached” to its own IF.

In analogous way to Newtonian physics, where Galilean transformations conserve the distance between points [i.e. \( dt'^2 \equiv dx^2 + dy^2 + dz^2 = \text{inv} \) and also similarly \( \nabla^2 = \text{inv} \) as we saw above] now in special relativity, we have a new “distance” in spacetime \( ds^2 \) which is invariant. To characterize this “distance” in spacetime between events it is useful to introduce the Minkowski metric tensor

\[
g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\]  \hspace{1cm} (31)

in terms of which we can write the invariant distance in spacetime as,

\[
ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu
\]  \hspace{1cm} (32)

The non-Euclidean character of the geometry of spacetime is obvious since the square of the invariant distance (the interval \( ds^2 \)) between different spacetime points (events) can be positive, zero or negative. But since \( \Delta s^2 \) is an invariant, all observers agree on its sign, so it makes sense to classify a pair of events according to the sign of their \( \Delta s^2 \):

- \( \Delta s^2 > 0 \) \( \Rightarrow \) timelike: there is an IF where these events happen at the same position \( (\Delta x' = \Delta y' = \Delta z' = 0) \)
- \( \Delta s^2 = 0 \) \( \Rightarrow \) light-like or null: events connected by light propagation
- \( \Delta s^2 < 0 \) \( \Rightarrow \): there is an IF where events happen at the same time \( (\Delta t' = 0) \), i.e. simultaneous
The boundary $\Delta s^2 = 0$ plays an important role in relativity as it separates events that can be simultaneous (spacelike) from events that can be connected by particles motion (timelike). All events relative to a given one e.g. $P$ that have $\Delta s^2 = 0$ form the lightcone at $P$.

$A$ & $C$ are inside the lightcone of $P$, $A$ is in the future lightcone, $C$ in the past lightcone. Events inside the lightcone are casually connected to $P$ since they are timelike separated ($\Delta s^2 > 0$) and can be connected by particle motion with speed less than $c$, i.e. $\Delta s^2 = c^2 \Delta t^2 - (\Delta \vec{r})^2 > 0$, $\Rightarrow |\Delta \vec{r}|^2 < c \Delta t$

On the contrary, $B$ is outside of $P$’s lightcone and thus causally disconnected from $P$ ($\Delta s^2 < 0$), one needs faster than light propagation to connect $B$ to $P$.

Clocks measure timelike distances, while rulers measure spacelike distances. Timelike distances that correspond to events that occur at the same point in space are special since $\Delta \vec{r} = 0$, $\Rightarrow \delta s^2 = c^2 \Delta t^2$ and they are invariant. These are known as proper time $\tau$.

$$\Delta s^2 = c^2 \Delta \tau^2$$ (33)

and is the time measured by a clock carried along the worldline of a particle.

We can now draw the different coordinates in the same spacetime diagram. We have

$$x' = \gamma (x - \beta t)$$ (34)  
$$ct' = \gamma (ct - \beta x)$$ (35)

The equation of motion for origin of $S'$ ($x' = 0$) is just $x = \beta ct$ or $c(\gamma t - \beta^{-1} x)$ (ct' axis) equivalent to a particle moving with velocity $V$: this is no other than the ct' axis (i.e. worldline of $x' = 0$).
Similarly, the $x'$ axis corresponds to all simultaneous events in $S'$ i.e. to $ct' = 0$ (so it is a spacelike line), which gives

$$ct = \beta x \quad (x' \text{ axis}) \quad (36)$$

Since they are inverse slopes ($\beta^{-1}$ and $\beta$), the $x'$ axis is a rotation of $x$ by the same angle ($\theta$) that $ct'$ is a rotation of $ct$ in the opposite direction!

Then it becomes obvious that simultaneous events in $S$ are not simultaneous in $S'$ and vice versa.

$$A \& B \text{ simultaneous in } S, \text{ but } B \text{ happens earlier in } S' \ (t' = 0) \text{ than } A \ (t' > 0). \text{ Also } A \& C \text{ are simultaneous in } S' \text{ but } C \text{ is later than } A \text{ in } S.$$

Now, disagreement about simultaneity means disagreement about length, since to measure the length of a moving object (say, a rod) we record its left and right coordinates $x'_1$ & $x'_2$ at the same time $t'$ to determine length $L' = x'_2 - x'_1$. Compare this with the length $L$ measured when the object is at rest in $S$: $L = x_2 - x_1$.

From LT’s we have:

$$x_2 = \gamma(x'_2 + \beta ct') \quad (37)$$
$$x_1 = \gamma(x'_1 + \beta ct') \quad (38)$$

$$\Rightarrow L' = \gamma^{-1}L \leq L \quad (39)$$

i.e. length is maximum when measured in object at rest (“proper length”. The length of an object in motion $L'$ is always less than proper length: this is known as Lorentz contraction.)
Similarly, we can now discuss time-dilation. Proper time $\tau$ of an object is the time measured by a clock carried along the worldline of the object, i.e. in the frame where the object is at rest:

\[ d\tau = \frac{ds}{c} = \frac{1}{c} \sqrt{c^2 dt^2 - dx^2 - dy^2 - dz^2} = dt \sqrt{1 - \frac{dx^2 + dy^2 + dz^2}{c^2 dt^2}} \]  

\[ \Rightarrow d\tau = dt \sqrt{1 - \frac{v^2}{c^2}} = \frac{dt}{\gamma} \Rightarrow d\tau \leq dt \]  

i.e. a clock that is moving (measuring $\tau$) “runs slow” compared with the clock that is at rest in system where we see object moving with velocity $v$. In other words, time in frame where events do not occur at same point in space (object is moving) has $dt = \gamma d\tau > d\tau$. This is known as time-dilation.

We can finally go back to the example given earlier about the two cylinders that show inconsistency of EM with GT’s and see how this is resolved with Lorentz transformations. We now do the transformation to the IF where $+$ charges are at rest by considering that relative to the original IF (left picture) in the new frame $-$ charges are Lorentz contracted (while the opposite is true for the $+$ charges):

Now in the new IF (right) negative charges are more densely packed than positive, so the upper cylinder is negative charged and thus this results is net electric attractive force, which as we shall see later is some magnitude as the $E_{\text{mag}}$ on original frame.