Renormalization Group: Real Space

Breakdown of Landau (and MFT) Approximation

Consider the average magnetization $M$ on a volume $V = \varepsilon^d$ of linear dimensions of size $\varepsilon$. We want to see the relative level of fluctuations to estimate the validity of MFT (this exercise is known as the Goldstone criterion). Since $M^2 = -\frac{6r_0}{\nu_0} = 6\frac{\varepsilon}{\nu_0}$ for $T < T_0$, we have

$$M^2 = M^2 V^2 = \frac{6\varepsilon}{\nu_0} (T_0 - T)^2 \varepsilon^{2d-2} \uparrow \varepsilon^{-d} \quad \text{above}$$

Above, for fluctuations, we have

$$\Delta M^2 = \int d^d x \delta^2 \Theta(x-y) = V \int d^d r \Theta(r) \int V \int \frac{d^d r}{r^{d-2}} \sim V^2 \varepsilon^{2d}$$

$$\Rightarrow \frac{\Delta M^2}{M^2} \sim \frac{V^2}{\frac{6\varepsilon}{\nu_0} (T_0 - T)^2 \varepsilon^{2d}} \sim \frac{\varepsilon^{2d}}{\varepsilon^{2d-2}} \sim \varepsilon^2$$

Thus, for $d > 4$, $\frac{\Delta M^2}{M^2} \to 0$, so as $T \to T_0^-$ fluctuations become small. However, for $d < 4$, $\frac{\Delta M^2}{M^2} \to \infty$ as we approach the critical point, thus Landau approximation breaks down.

Note: even if $d < 4$, it may be still possible to find a temperature range $\Delta T$ around $T_0^-$ such that $\frac{\Delta M^2}{M^2} < 1$ if $T \in [T_0 - \Delta T, T_0]$. For example, in high-conductivity systems or near a phase transition $\Delta T$ is so small that the region where Landau theory fails is invisible in practice.

If one calculates corrections to the Landau theory by going beyond saddle-point evaluation of partition functions, one finds that corrections are also proportional to $\varepsilon^{2d-4}$, thus perturbation theory itself breaks down as we get close to the critical point. The problem is that as $T \to T_0^-$ the number of degrees of freedom interacting with each other is of the
The Renormalization Group (RG): Main Idea

The RG approach has two main objectives,

1. To simplify the task of solving systems with many degrees of freedom contained within a correlation length.

The analogy here is hydrodynamics, where one introduces new macroscopic variables, such as density, which represents an average over the microscopic degree of freedom. Thus, there are far fewer hydrodynamic degrees of freedom per unit volume, compared to the original microscopic system.

The RG approach is similar to hydro in the sense that original microscopic degrees of freedom are replaced by a smaller set of effective degrees of freedom. In the RG, the reduction of the density of degrees of freedom is carried out iteratively, by averaging over small-scale fluctuations. We will discuss this in some detail below, but the idea is that after one step of the RG transformation, one has now degrees of freedom "separated" by distance 2\(a\) (where \(a\) is the original lattice spacing), and the factor of 2 could be some other factor (even infinitesimal, continuous, as we shall see later in Fourier approach).

The new Hamiltonian of these degrees of freedom is \(H_2\), that can be found from original \(H\) after averaging process. In general, \(H_2\) will have different coupling constants (in amplitude and number of them) from \(H\), but the idea is that the effective interactions are still local, i.e. spins couple mainly to nearby spins. If one repeats this process iterating eventually after \(n\) RG's so that \(2^n a = \xi\), one has effectively a small number of degrees of freedom per correlation length, the critical behavior can be solved by perturbative (or other) methods, and then if one can relate it to critical behavior of original system, the problem is solved.
Dramatically, 

\[ H_0 \rightarrow H_1(a) \rightarrow H_2(2a) \rightarrow \ldots \rightarrow H_n(2^na) \]

one has some amusing procedure that goes from dof separated by 

lattice spacing \( a \) to \( 2a \), thus induces a new Hamiltonian \( H_2 \).

The point is that calculating \( H_n \) from \( H_{n-1} \) only requires treating 

a small subset of dof since these are thought to be local 

interactions, so only dof between \( 2^{n-1}a \) and \( 2^na \) need to be 

calculated. On the other hand, since we are averaging over 

small scales compared to correlation length (at least before \( 2^n a < \xi \)) 

the correct critical behavior of these systems should be the 

same—Thus, correlation functions between original system and 

renormalized system can be mapped, and if renormalized can 

be estimated by perturbation theory, then we overcome the 

problem of dealing with a large number of dof.

III) To explain "universality", i.e. why systems with different Hamiltonians 

have the same critical behavior (same critical exponents).

The way this arises in R6 is roughly as follows. Let \( T \) be the R6 T 

that maps the Hamiltonians, i.e. 

\[ T(H_0) = H_1, \quad T(H_1) = H_2, \ldots, T(H_n) = H_n \]

Since one applies a large number of times so that \( 2^na > \xi \), the 
simplest result one can obtain (though not guaranteed, of course) is that 

for large \( n \) this sequence of Hamiltonians approaches a fixed point 

\( H^* \) of \( T \), i.e. 

\[ T(H^*) = H^* \]

Note this is a property of the transformation \( T \) itself, and makes no 

reference to the original Hamiltonian \( H_0 \). The possible types of critical 

behavior, in the R6 picture, are determined by the fixed points of \( T \). If 

we have more than one fixed point, say \( H_{1a}^* \) and \( H_{1b}^* \), one has to 

construct the sequence \( H_1, H_2, \ldots \) to see which fixed point is the
relevant one for $H_0$. Thus, the set of all possible interactions $H_0$ will be divided into 2 domains, one for each fixed point. All Hamiltonians within a domain will flow to, say $H_0^*$, and their critical behavior will be identical.

**RG: 1D Example**

We already showed how to solve the 1D Ising model exactly - let us now do a RG for this case. We need to illustrate how things work. We will perform a very simple averaging, called decimation, in which (by) spins are traced over at a time, leaving a new lattice with untraced spins separated by distance $b$ ($b=4$ in this figure)

![Decimation Diagram]

Now, let's carry over this procedure for $b=2$, for simplicity. Remember, the idea is to figure out, after the averaging procedure, what is the new Hamiltonian in terms of the old one. Let's start with the following Ising Hamiltonian:

$$-\beta H = +K \sum_i S_i S_{i+1} + \sum_i K_0$$

Note that we introduce a constant term $K_0$, hence the reason for this will become clear soon (the RG introduces such constant after iteration, even if we start with $K_0 = 0$). Such a constant just redefines the zero of the free energy. The nice thing about decimation procedure is that it will keep the same structure of the Hamiltonian, i.e., after this procedure the coupling is still nearest neighbor. Thus, all we have to do is to figure out $K_1^*$ and $K_0^*$ in order to characterize $H_1$ (or $H^*$) for the new degrees of freedom.

Let's assume $b=2$, so we trace over one intermediate spin only. Clearly,
we can do this one spin at a time to derive coupling between neighboring spins in new lattice. By definition, new coupling between \( s_1 \) and \( s_3 \) is

\[
e^{k' s_1 s_3 + k_0'} = \sum_{n=0}^{\infty} \frac{e^{k (s_1 s_2 + s_2 s_3)} e^{2 k_0}}{2 \cosh[k (s_1 + s_3)]} = e^{k_0' \cosh[k (s_1 + s_3)]}
\]

This is important: is the statement we made above that to figure out \( H_4 \) from \( H_0 \) we only need to take into account a small number of dof at a time.

\[
\Rightarrow k' s_1 s_3 + k_0' = 2 k_0 + \ln 2 + \ln \cosh[k (s_1 + s_3)]
\]

Now \( \ln \cosh[k (s_1 + s_3)] = \sum_{s=0}^{\infty} \ln \cosh 2k s_1 = s_3 \)

\[
\Rightarrow k' s_1 s_3 + k_0' = 2 k_0 + \ln 2 + \frac{(1+s_1 s_3)}{2} \ln \cosh 2k
\]

\[
\Rightarrow k' = \frac{1}{2} \ln \cosh 2k
\]

\[
k_0' = 2k_0 + \ln 2 + k'
\]

Now, let's look at \( k' \) as a function of \( k \):

![Graph showing the relationship between \( k' \) and \( k \)]

We see that always \( k' < k \), and we see that as we iterate the RST, \( k' \) is driven to zero! In fact, \( k=0 \) is a fixed point, since

\[
k' = \frac{1}{2} \ln \cosh 2k = 0 \]

as well. To calculate the partition function using renormalization, note that by definition of decimation procedure we have

\[
\tilde{Z}(k', k_0) = \sum_{s_1, s_2, s_3, \ldots} e^{k' (s_1 s_2 + s_2 s_3 + \ldots)} e^{N k_0} = \tilde{Z}(k, k_0) = \sum_{s_1, s_2, s_3, \ldots} e^{k (s_1 s_2 + s_2 s_3 + \ldots)} e^{N k_0} = \tilde{Z}(k, k_0)
\]

for odd \( N \) spins.
Thus, we can calculate the partition function of the original system using the renormalized system. However, in the original system, the coupling constant is not necessarily small, thus calculation may not be possible; whereas in the renormalized system, if we iterate RG enough, the coupling $k'$ will be driven to zero and the partition function becomes that of a non-interacting system, where $k_0$ provides the only contribution.

In 1D Ising case, we know the answer already because we can solve it exactly, i.e.,

$$g = n \cosh k$$

for large $N$ (when $k_0 = 0$). Thus, if our system has $k = 3$ and $k_0 = 0$, the free energy per site is

$$-\beta f(k, k_0 = 0) = \frac{\ln 2}{N} \ln 2 + \ln \cosh k = -3.00248$$

This is the exact result. The calculation by RG says that

$$-\beta f(k, k_0 = 0) = -\frac{1}{2^n} \beta f(k', k_0') = -\frac{1}{2^n} \beta f(k'', k_0'') = \ldots = -\frac{1}{2^n} \beta f(k^{(m)}, k_0^{(m)})$$

since at each step we halve the number of spins.

But now, as $n \to \infty$, the coupling $k^{(m)} \to 0$, so we can just use perturbation theory to evaluate the free energy, which in this limit has the value

$$-\beta f(k_0) = \ln 2 + k_0$$

$$\zeta = e^{-\ln 2} 2^n$$

Then, at each iteration we can approximate the free energy per site as follows

$$-\beta f_{\text{approx}} = -\frac{1}{2^n} \beta f(k^{(m)}, k_0^{(m)}) \approx \frac{1}{2^n} \beta f(k_0^{(m)}) = \frac{1}{2^n} \left( k_0^{(m)} + \ln 2 \right)$$

(Neglect $k^{(m)}$)
As we iterate, $t^{(m)}$ decreases and soon no longer contributes to the free energy, so most of it is given by the trivial (non-interacting) term $K_0^{(m)}$, and as $n$ becomes large we get the right answer [show plot of RG iterations]. Basically, with only 4 iterations we will get the free energy per site to better than 10%. - See plot at the end.

In fact, the RG iteration for the nearest neighbor coupling constant can be solved exactly. Note that

$$e^{2k'} = \cosh 2k = \frac{e^{2k} - e^{-2k}}{2}$$

Now:

$$\tanh k' = \frac{e^{k'} - e^{-k'}}{e^{k'} + e^{-k'}} = \frac{e^{2k'} - 1}{e^{2k'} + 1} = \frac{e^{2k} - e^{-2k}}{e^{2k} + e^{-2k}} = \left(\frac{e^k + e^{-k}}{e^k - e^{-k}}\right)^2$$

$$\Rightarrow \tanh k' = (\tanh k)^2$$

After $n$ steps:

$$\tanh k^{(n)} = (\tanh k)^2^n$$

Now, we have done $b=2$ in each step. Two steps with $b=2$ is equivalent to one step with $b=4$, in general $n$ steps of $b=2$ corresponds to decimation in one step with $b=2^n$, then for arbitrary $b$ we have that in each step

$$\tanh k' = (\tanh k)^b$$

This makes sense from the point of view of the exact solution of the 1D Ising model. Remember that the two-point function $G_{ij}$ was

$$G_{ij} = e^{-\frac{a}{2}} = e^{-\frac{1}{2} - 1} \frac{a}{2} = (\tanh k)^{i-j}$$

$$\Rightarrow e^{-\frac{a}{2}} = \tanh k \quad \Rightarrow \quad \frac{a}{2} = \frac{1}{\ln \tanh k}$$

Now, since the renormalized model is also an Ising model ($K_0$ does not enter into correlation functions) we must have

$$\frac{g_1}{2a} = \frac{1}{\ln \tanh k} \quad \Rightarrow \quad \frac{1}{2a} = \frac{1}{\ln \tanh k} = \frac{1}{2 \left(\frac{3}{2}\right)}$$

$a$ new lattice spacing
\[ \Rightarrow \text{we have} \quad \bar{\xi}^\prime = \frac{\bar{\xi}}{2} , \]

as expected, tracing over small scale does not change large-scale behavior.

In terms of the correlation length in units of the lattice spacing we have,

\[ \bar{\xi}^\prime = \frac{\bar{\xi}^\prime}{2a} = \frac{\bar{\xi}}{2a} = \frac{1}{2} \bar{\xi} \]

Due, after n iterations of RG,

\[ \bar{\xi}^{(n)} = \frac{\bar{\xi}}{2^n} \quad \text{or} \quad \bar{\xi}^{(n)} = \frac{\bar{\xi}}{b} \]

that is, the number of dof inside a correlation length decreases exponentially as we iterate.

Now, the RG recursion relation

\[ \tanh k^\prime = (\tanh k)^b \]

has two fixed points, \( \tanh k^\prime = (\tanh k)^b \)

\[ \begin{align*}
\tanh k^\prime &= 0 \quad (T = \infty) \quad (\bar{\xi} = 0) \\
\tanh k^\prime &= 1 \quad (T = 0) \quad (\bar{\xi} = \infty)
\end{align*} \]

As we saw, the first fixed point is stable, i.e. attracts all initial \( \bar{\xi} \)'s that are finite. The other fixed point, \( \bar{\xi} = \infty \), is unstable and corresponds to the zero temperature critical point (at which there is a phase transition, except it is kind of irrelevant in practice since we only care about \( T \) to phase transitions).

The phase diagram thus looks as follows,

\[ \begin{array}{c}
\text{weak coupling fixed point} \quad (\text{single}) \quad \begin{array}{c}
\bar{\xi} = 0 \\
T = 0
\end{array} \\
\text{critical fixed point} \quad (\text{unstable}) \quad \begin{array}{c}
\bar{\xi} = \infty, \tanh k = 1 \\
T = 0
\end{array}
\end{array} \]

Beyond 1D: Migdal (1973) - Kadanoff (1976) - Bord-Wiese scheme

OK, let's go beyond 1D which is too special to be of any practical interest.
Choosing the averaging procedure beyond 2D turns out to be tricky, remember a nice thing about decimation in 1D is that it generated a renormalized Hamiltonian with the same structure (just two couplings $t_0, t_1$) as we iterate. In 2D, decimation leads to a hopelessly complicated Hamiltonian as we iterate, generating beyond nearest neighbor interactions, increasingly so as we iterate.

Migdal and Kadanoff figured out a way of improving this situation by combining decimation with a bond moving scheme; this is not a fully controlled approximation scheme, but it is rather reasonable and thus useful to illustrate what is going on.

Suppose we work in 2D, and we start with a nearest neighbor Ising model - we have a spin lattice like:

![Spin lattice diagram]

Suppose we want to trace over the crosses. The problem we have now, that we didn't have in 1D, is that the trace is not over independent spins, because although they are independent along $x$-direction, along $y$-direction the trace involves coupled spins - [If you try something like $\star \star \star$, the problem is that leads to next nearest neighbor interaction as mentioned above]. Thus, the trick in the bond moving scheme is to "move" the couplings in the $y$-direction to the $\star$ spins, then perform a trace once couplings are gone. Then the process is repeated with $x$ and $y$ interchanged in an attempt to restore
Symmetry between $x$ and $y$ couplings - Schematically,

$$
\text{\textit{(lines denote couplings)}}
$$

let $R^b(K)$ denote the operator defined by decimation

in 1D, i.e. $k' = R^b(K) = \text{Ann} \text{Tanh}[\text{Tanh}(K)^b]$,

Then we have the following procedure,

$$
k_x' = b R^b(k_x) \quad k_y' = b k_y
$$

Thus, after one iteration, we get

$$
\begin{cases}
    k_x' = b R^b(k_x) \\
    k_y' = R^b(k_y)
\end{cases}
$$

If we do this in $d$ dimensions, we have $K^b$ ($k_x = K_x$, $k_y = K_y$, etc up to $K_d$), and the general recursion relation for a hypercubical lattice is:

$$
k_i^b = b^{d-i} R^b(b^{d-i} K_i)
$$

To illustrate how these recursion relations describe a non-trivial cut-off point at finite temperature, we will consider the limit in which $b \Delta$ is infinitesimal, $b = e^{\delta b}$ with $\delta b$ small. Although in practice this is not doable in a lattice unless $b$ is integer, the recursion relation can be thought as valid for arbitrary $b$ by analytic continuation to all $b$ real. Then we have to first order in $\delta b$:

$$
k_i^b = K_i + \delta b \left[ K_i (d-1) + \sinh(K_i) \cosh(K_i) \ln \text{Tanh}(K_i) \right] + O(\delta b^2)
$$

Note that to this order, things are independent of $i$, so all couplings
will transform the same way. Let $K_i = K$, then the infinitesimal change in $K$ is given by:

$$\frac{dK}{d\tau} = (d-1)K + \sinh K \cos K \ln \cosh K$$

Now, we note that for $d=1$, there is a new term absent in $d=1$ (which gives the same recursion relation as before). Since the second term goes to $-\frac{1}{2}$ for large $K$, the first term will dominate as $K$ gets large. To explore what's going on at large $K$, look at $T = \frac{1}{K}$

$$-\frac{1}{T^2} \frac{dT}{d\tau} = (d-1) \frac{1}{T} - \frac{1}{2} \Rightarrow \frac{dT}{d\tau} = -\epsilon T + \frac{1}{2} T^2$$

Now, the fixed points of this equation, when coupling does not change, are:

$$T^* = 0 \quad T^* = 2\epsilon$$

The first one is the usual zero-temperature fixed point ($K=0$), the second one is a new fixed point. There is also the $T=0$ fixed point ($K=0$). By expanding about $T^*$, we can check stability for $T = T^* + \delta T$

$$\frac{d\delta T}{d\tau} = (-\epsilon + T^*) \delta T$$

Thus, for $T^* = 0$:

$$\frac{d\delta T}{d\tau} = -\epsilon \delta T < 0 \quad \text{so} \quad T^* = 0 \text{ is stable, whereas the}$$

$$T^* = 2\epsilon$$

$$\frac{d\delta T}{d\tau} = \epsilon \delta T > 0$$

so it is unstable, perturbations away from $T^*$ grow. (the fixed point @ $K=0$ is also stable). Therefore we have the following picture:

```
Weak coupling fixed point
K=0 ----> Critical fixed point
K*K
(T*=2\epsilon)
---\rightarrow Order phase
(F=0)
```

Strong coupling fixed point
Now let's see how the correlation length exponent follows from these recursion relations. For $T$ close to $T_{c+}$, we have

$$\frac{\text{d} \delta T}{\text{d} b} = \epsilon \delta T \implies \delta T(b) = e^{\epsilon b} \delta T(0)$$

On the other hand, the correlation length change, as we iterate the RG transformation,

$$\frac{\xi(b)}{\xi(0)} = \frac{1}{b} = e^{-\delta b} \equiv \left[ \frac{\delta T(b)}{\delta T(0)} \right]^{-\nu} = e^{-\nu \delta b}$$

Then, we can read off the critical exponent $\nu$,

$$\nu = \frac{d}{\epsilon} = \frac{1}{d-1}$$

Note that if blindly use $d=3$ ($\epsilon=1$), we recover the exact result in 2D, $\nu=\frac{1}{2}$, even though we were assuming $\epsilon$ is small. From this simple example we see how linearization of the RG transformation about the critical fixed point gives the critical exponents. We shall explore this more later.