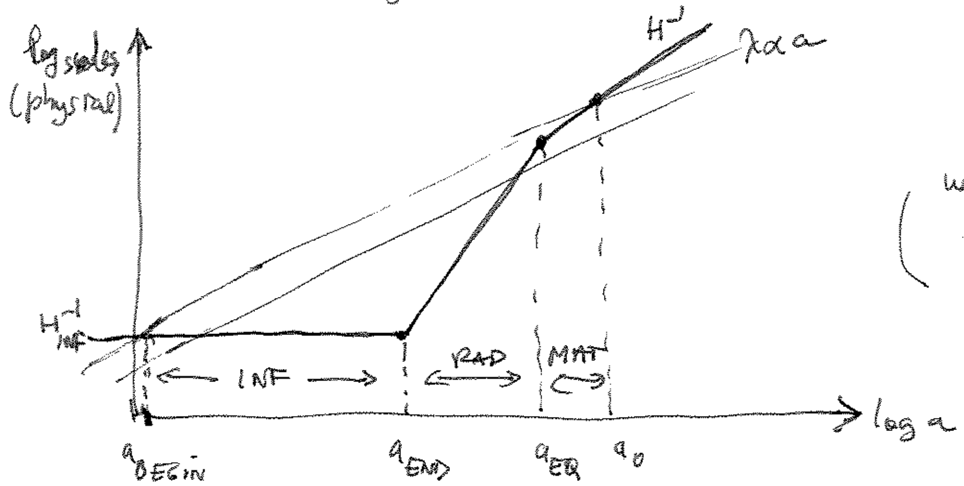
FLUCTUATIONS FROM INFLATION

We now discuss how inflation generate density (and gravity wave) perturbations. The goal is to predict, given a model of inflation, the statistical properties of the fluctuation, i.e. whether they are Gaussian, and what is their power spectrum (amplitude, spectral index), etc. These predictions are very important since they can be tested against observations we make now. Once we have the properties of fluctuations generated during inflation, we will evolve them until the present to make that connection with present observations. That will take the rest of this course.

To start the discussion, let's recall the basic picture for the evolution of scales when inflationary epoch is added to the standard big-bang model:



(we are ignoring reheating phase in this plot, for simplicity)

The idea behind inflation is that scales are inside the Hubble radius during inflation. We can estimate the physical size of perturbation wavelengths ~~relative~~ that correspond today to our Hubble radius by noting that H_0^{-1} today corresponds to a physical size at least H^{-1}_{INF}, then

$$H^{-1}_{INF} = \frac{\sqrt{3}\, M_{pl}}{\sqrt{V_0}} \sim \frac{10^{19}\,\mathrm{GeV}}{\sqrt{(10^{16}\,\mathrm{GeV})^4}} \sim 10^{-13}\,\mathrm{GeV}^{-1} \sim 10^{-27}\,\mathrm{cm}\ !$$

$H^2 = \frac{V_0}{3M_{pl}^2}$ GUT scale $\mathrm{GeV}^{-1} \sim 2\times 10^{-14}\,\mathrm{cm}$

Thus, the scales relevant during inflation are so small that quantum effects can become important. Indeed, as we discuss now, quantum fluctuations is the mechanism by which inflation generates density perturbations.

To start the discussion of fluctuations we go back to the evolution of a scalar field ϕ in an FRW model:

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2}\nabla^2\phi + V'(\phi) = 0$$

where now we put back gradient terms as they are non-negligible for perturbations. Since we consider slow-roll inflation $V'(\phi)$ is negligible, and Fourier transforming and decomposing $\phi(\vec{x},t) = \bar{\phi}(t) + \delta\phi(\vec{x},t)$ we have:

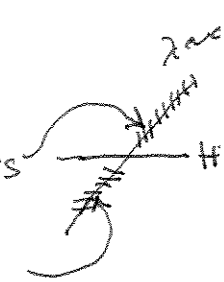
$\bar{\phi}(t)$: homogeneous part that we considered before; $\delta\phi(\vec{x},t)$: fluctuations

$$\ddot{\delta\phi}(\vec{k}) + 3H\,\dot{\delta\phi}(\vec{k}) + \frac{k^2}{a^2}\,\delta\phi(\vec{k}) \simeq 0$$

Naturally two length-scales appear in the problem. One is the Hubble radius during inflation, H^{-1}. The other is the wavenumber of fluctuations (or inverse wavelength), which in physical units is $k_{phys} = \frac{k}{a}$ (recall that $\lambda_{phys} = \lambda_{comov}\, a$, this is just the same relationship after using $\lambda = \frac{2\pi}{k}$).

Therefore, the relevant ratio is the dimensionless quantity:

$$\underset{\text{(or wave vector)}}{\text{Inverse wavelength}} \times \text{Hubble radius} = \frac{k}{a}H^{-1} = \frac{k}{aH}$$



For $\frac{k}{aH} < 1$ perturbations are outside the Hubble radius

" $\frac{k}{aH} > 1$ " " inside " " "

Now, lets look at the equation of motion in these two limits:

When $\frac{k}{aH} \gg 1$, we are well inside the Hubble radius, and since $\dot{\delta\phi} \sim H\,\delta\phi$ we have approximately

$$\ddot{\delta\phi}_{(\vec{k})} + \left(\frac{k}{a}\right)^2 \delta\phi_{(\vec{k})} \approx 0 \qquad \left(\frac{k}{aH} \gg 1\right)$$

i.e. we recover a simple harmonic oscillator (recall $\frac{k}{a} = k_{phys}$)
This makes sense because at such small scales the expansion of the universe should not be important (apart from maping $k_{phys} = \frac{k}{a}$) and one should recover the Minkowski space result.

At large scales, on the other hand, the amplitude of such oscillator becomes frozen, for $\frac{k}{aH} \ll 1$ we have

$$\ddot{\delta\phi}(\vec{k}) + 3H\,\dot{\delta\phi}(\vec{k}) \approx 0 \quad \Rightarrow \quad \delta\phi(\vec{k}) \simeq \text{const.}$$

(the other indep solution is a decaying mode, irrelevant)

These two limits give us the basic behavior of inflationary perturbations: as perturbations are inside the Hubble radius they behave like a simple harmonic oscillator with a frequency that adiabatically adjusts to the evolution of $a(t)$ (through $k_{phys} = \frac{k}{a}$), as the perturbation crosses the Hubble radius the amplitude becomes frozen to a constant.

To discuss quantum fluctuations, and motivated by the above results, we rewrite a Lagrangian for these equations of motion for $\delta\phi(\vec{k})$ in the form of a harmonic oscillator (for each independent mode $\vec{k}$):

$$\mathcal{L}_k = \frac{a^3}{2}\left[\,|\dot{\delta\phi}(\vec{k})|^2 - \frac{k^2}{a^2}\,|\delta\phi(\vec{k})|^2\right]$$

which recovers through $\frac{d}{dt}\left(\frac{\partial \mathcal{L}_k}{\partial \dot{\delta\phi}(\vec{k})}\right) - \frac{\partial \mathcal{L}}{\partial \delta\phi(\vec{k})} = 0$ the equation of motion $\ddot{\delta\phi}(\vec{k}) + 3H\,\dot{\delta\phi}(\vec{k}) + \frac{k^2}{a^2}\,\delta\phi(\vec{k}) = 0$, as you can easily check.
This has the form of a simple harmonic oscillator with a

time-dependent mass:

$$\mathcal{L}_k^{HO} = \frac{m}{2}\left(\dot{x}^2 - w^2 x^2\right)$$

where $m \to a^3$, $x \to |\delta\phi(k)|$, $w = \frac{k}{a}$ gives the mapping from the HO to inflationary perturbations.

Now we can use our knowledge of the quantization of the harmonic oscillator. In the vacuum state (ground state), we have that fluctuations are Gaussian (as you will show in the homework!) and have a variance

$$\langle x^2 \rangle = \frac{\hbar}{m w}$$

which translating back to our problem means that the power spectrum of perturbations is

$$P_\phi(k) \sim \frac{\hbar}{a^3 (k/a)}$$

Now, let's follow what happens as a function of time with this. A given k-mode initially is smaller than the Hubble radius, doesn't see the expansion of the universe and just adjusts adiabatically,

$$P_\phi(k) \sim \frac{1}{a^2 k} \sim \frac{1}{k} \qquad \left(\frac{k}{aH} \gg 1\right)$$

However, once a mode crosses the Hubble radius, the fluctuations get frozen at a value

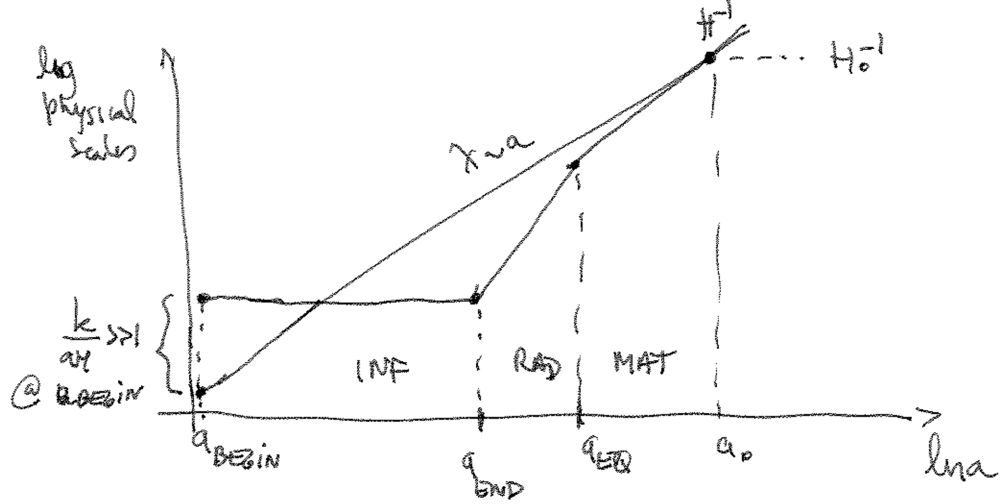
$$P_\phi(k) \sim \frac{\hbar}{a_*^3 (k/a_*)} \quad \text{where} \quad k = a_* H \qquad \text{(Hubble radius crossing)}$$

$$\Rightarrow \quad P_\phi(k) \sim \hbar \frac{H^2}{k^3} \sim \frac{1}{k^3} \qquad \left(\frac{k}{aH} \lesssim 1\right)$$

Therefore, we see that the spectrum of fluctuation changes shape, from $P_\phi \sim k^{-1}$ characteristic of Minkowski, to $P_\phi \sim k^{-3}$ as modes exit the Hubble radius. This is very important, as we shall discuss in a second.

In this discussion we have assumed that quantum fluctuations are in the vacuum state, why? The basic reason is that if there were substantial occupation numbers that automatically translates into significant energy density in the perturbations, and the equation of state would not be $p \simeq -\rho$ and inflation will not happen (recall $\rho \simeq \frac{\dot\phi^2}{2} + V(\phi) + \frac{|\nabla\phi|^2}{2a^2}$, $p = \frac{\dot\phi^2}{2} - V(\phi) - \frac{|\nabla\phi|^2}{6a^2}$, gradient dominance gives $p = -\frac{1}{3}\rho \Rightarrow \ddot a = 0$).

The ~~[illegible]~~ condition is $\left(\frac{k}{a}\right)^4 \ll V \sim 3 M_{pl}^2 H^2 \Rightarrow \frac{k}{aH} \ll \sqrt{\frac{M_{pl}}{H}} \sim \frac{M_{pl}}{V^{1/4}}$ $\sim 10^3$ for typical numbers ($V^{1/4} \sim 10^{16}$ GeV). Then if inflation starts much earlier than when the scale corresponding to our H_0^{-1} crosses the horizon during inflation, these modes are so dangerous ($\frac{k}{aH} \gtrsim 10^3$) that they must be in the vacuum state to avoid spoiling inflation. Pictorially:



Now, going back to the power spectrum we derived above, we see that after modes exit the Hubble radius, thus have

$$P_\phi(k) \sim \frac{H^2}{k^3} \sim k^{-3}$$

which means that the amplitude of fluctuations after Hubble crossing is

$$4\pi k^3 P_\phi(k) \sim H^2 \sim \text{constant}$$

since H is approx. constant during inflation. Therefore, inflation generates amplitude of fluctuations that are the same for all scales, this is known as the "scale invariant" spectrum, and was obtained by heuristic arguments (unrelated to inflation) by Harrison & Zel'dovich, so it is usually referred to as the Harrison-Zeldovich spectrum as well.

This is very important, as the shape of the spectrum gets preserved until it crosses back inside the Hubble radius much later during RAD and MAT eras, so it can be tested. Observations agree very well with this basic property of inflationary perturbations (we will discuss this in much more detail later).

Another important consequence of inflation is that it also generates a stochastic background of gravitational waves. This is so because gravitational waves can be thought as free massless scalar fields (one for each polarization). Gravitational waves are called <u>tensor</u> perturbations (corresponding to tensor modes of metric perturbations) as opposed to the <u>scalar</u> perturbations that are related to $\delta\phi_k$. This basic prediction of inflation has not yet been detected (we shall say more about this next class).

One last comment. ~~why do we need inflation if there are quantum~~
One may wonder why do we need inflation, i.e. is it possible to take advantage of quantum fluctuations in some other way?
First thing one should realize is that without ~~quan~~ inflationary stretching, quantum fluctuations are hopelessly small at, say, galactic scales, since $P_\phi \sim \frac{1}{a^2 k}$ the amplitude of fluctuations is (without inflation)

$$\Delta_\phi \sim 4\pi\, k^3 P_\phi \sim \frac{1}{a^2}\, k^2$$

whereas this number is of order unity for the planck length $k_{planck} \sim \frac{2\pi}{\ell_{planck}} \sim (10^{-33} cm)^{-1}$ it goes down as $\frac{1}{\lambda^2}$ so at galactic scales $\lambda_{gal} \sim 10^{25} cm$: $\Delta\phi \sim \left(\frac{10^{-33}}{10^{25}}\right)^2 \sim 10^{-120}$!

The other issue is that ~~as the universe expands~~ the spectrum would be incorrect (not having $\simeq$ equal power in all scales)

We see that inflation takes care of these two problems rather nicely: as the universe expands the power goes down as $\frac{1}{a^2}$ and fluctuations are stretched from tiny to large scales. At the same time, not all scales loose amplitude in the same way: after a scale crosses the Hubble radius its amplitude gets frozen (avoiding decaying to tiny values), as it gets frozen it continues to be stretched to macroscopic scales, and the shape of the spectrum gets changed from $P_\phi \sim k^{-1}$ to $P_\phi \sim k^{-3}$ for super-Hubble modes, as ~~distant~~ smaller-scale modes experience more time loosing amplitude than larger-scale (lower-k) modes. Schematically, the time evolution of the spectrum is like: