Inflationary Observables

Here we will try to summarize how we go from predictions to some thing close to observables. The first thing to consider is that perturbations in the scalar field must be transferred into something else to become observable since the scalar field decays at the end of inflation and is not around today.
We can relate the perturbations in $\delta\phi$ to perturbations in its energy density,

$$\rho_\phi \sim V \quad \Rightarrow \quad \delta\rho_\phi \sim \frac{\partial V}{\partial\phi}\,\delta\phi \underset{\underset{3H\dot\phi + V' \sim 0}{\uparrow}}{\sim} -3H\dot\phi\,\delta\phi$$

these energy density fluctuations can be shown (using GR) to lead to curvature perturbations R through

$$R \sim \frac{\delta\rho}{\rho+p} \underset{\uparrow}{\simeq} \frac{-3H\dot\phi\,\delta\phi}{\dot\phi^2} = -\frac{H}{\dot\phi}\,\delta\phi$$

$$\rho = \dot\phi^2/2 + V$$
$$p = \dot\phi^2/2 - V$$

And thus the power spectrum of curvature perturbations is (putting all numerical factors together)

$$\Delta_R(k) = 4\pi k^3 P_R(k) = \left(\frac{H}{\dot\phi}\right)^2\left(\frac{H}{2\pi}\right)^2\Bigg|_{\left(\frac{k}{aH}=1\right)}$$

: evaluated at Hubble radius crossing when perturbations are frozen

The important point of why we use R is that it exists well after inflation is over and ϕ is gone, in addition it has nice properties when $\frac{k}{aH} < 1$ which we'll discuss later, and when perturbations come back inside the Hubble radius again we can translate curvature perturbations to gravitational potential fluctuations,

$$R_k \propto \Phi_k \qquad \Phi_k \text{ is grav. potential (not } \phi\text{, the inflaton!)}$$

where the constant of proportionality depends on whether the k mode crosses H^{-1} when the universe is RAD or MAT dominated.

Because of the Poisson equation,

$$\nabla^2 \Phi = 4\pi G \rho = 4\pi G \bar{\rho}\,(1+\delta) \quad \Rightarrow \quad -k^2 \Phi_k \propto \delta_k$$

where δ are the density fluctuations, $\delta \equiv \frac{\rho - \bar{\rho}}{\bar{\rho}}$, therefore

$$k^4 P_\Phi \propto P_\delta(k)$$

$$\Rightarrow \boxed{P_\delta(k) \propto k^4 P_\Phi \propto k^4 P_R \propto k^4 \times k^{-3} \sim k^{+1}}$$

This tells us how the power spectrum of density perturbations is expected to scale at large scales, with spectral index equal to 1 for scale-inv. case.

Now, let's go back to inflation and work out some important observables.

We can rewrite the scalar curvature perturbations as

$$\Delta_R(k) = \frac{H^2}{\dot{\phi}^2}\left(\frac{H}{2\pi}\right)^2 \underset{3H\dot{\phi} \simeq -V'}{=} 9\,\frac{H^4}{(V')^2}\left(\frac{H}{2\pi}\right)^2 \underset{H^2 = \frac{V}{3M_{pl}^2}}{=} \frac{9}{4\pi^2}\,\frac{V^3}{27 M_{pl}^6}\,\frac{1}{(V')^2} = \frac{1}{12\pi^2 M_{pl}^6}\,\frac{V^3}{(V')^2}$$

using that the slow-roll parameter $\varepsilon = \frac{1}{2} M_{pl}^2 \left(\frac{V'}{V}\right)^2$ we get

$$\boxed{\Delta_R(k) = \frac{1}{24\pi^2 M_{pl}^4}\,\frac{V}{\varepsilon}}$$

define the scalar spectral index n_s by:

$$n_s(k) - 1 \equiv \frac{d \ln \Delta_R(k)}{d \ln k}$$

This may look like a strange definition, but recall that since

$$\Delta_R \sim k^3 P_R \sim k^3 \frac{P_\delta}{k^4} \Rightarrow \frac{d \ln \Delta_R}{d \ln k} = \frac{d \ln P_\delta}{d \ln k} - 1$$

$$\Rightarrow n_s = \frac{d \ln P_\delta}{d \ln k},$$ then n_s is the spectral index of density perturbations (when perturbations cross Hubble back)

In order to ~~evaluate~~ evaluate $n_s(k)$ for a given model of inflation (3)
we need to relate a change in k to a change in ϕ during inflation.
Since the spectrum is evaluated @ H^{-1} crossing $k = aH$ we have

$$dk = H\,da = H\,\dot{a}\,dt \qquad \text{since } H \simeq \text{const. during inflation}$$

$$\Rightarrow d\ln k = \frac{dk}{k} = H\,dt \underset{\substack{\uparrow \\ -3H\dot\phi \simeq V'}}{=} -\frac{3H^2}{V'}\,d\phi$$

Since $H^2 \equiv V/3M_{pl}^2$ we have : $d\ln k = -\frac{V}{M_{pl}^2 V'}\,d\phi \Rightarrow \frac{d}{d\ln k} = -M_{pl}^2 \frac{V'}{V}\frac{d}{d\phi}$

Then we need

$$n_s(k) - 1 = \frac{d\ln \Delta_R}{d\ln k} = \frac{d\ln V}{d\ln k} - \frac{d\ln \varepsilon}{d\ln k}$$

Now: $$\boxed{\frac{d\ln\varepsilon}{d\ln k}} = \frac{1}{\varepsilon}(-M_{pl}^2)\frac{V'}{V}\frac{d}{d\phi}\varepsilon = -M_{pl}^2\frac{V}{V'}\frac{d}{d\phi}\left(\frac{V'}{V}\right)^2$$

$$= -M_{pl}^2\frac{V}{V'}\left[\frac{2V'V''}{V^2} - 2\frac{V'^3}{V^3}\right] = -2M_{pl}^2\frac{V''}{V} + 2M_{pl}^2\left(\frac{V'}{V}\right)^2 \boxed{= -2\eta + 4\varepsilon}$$

$$\frac{d\ln V}{d\ln k} = -M_{pl}^2\frac{V'}{V}\frac{d}{d\phi}\ln V = -2\varepsilon$$

$$\Rightarrow \boxed{n_s(k) - 1 = 2\eta - 6\varepsilon}$$

this says that deviation from "scale-inv" are related to slow roll parameters η, ε, so they are expected to be small

We can do the same for the gravitational wave spectrum (or tensor spectrum)

$$\Delta_{GW}(k) = \frac{2}{M_{pl}^2}\left(\frac{H}{2\pi}\right)^2\Big]_{k=aH}$$

define a tensor spectral index $\boxed{n_T \equiv \frac{d\ln\Delta_{GW}}{d\ln k} = \frac{d\ln V}{d\ln k} = -2\varepsilon}$

Now, by comparing power spectra and spectral indices we can derive a "consistency relation". Indeed:

$$\Delta_R = \frac{1}{4\varepsilon}\Delta_{GW} \quad \Rightarrow \quad \frac{\Delta_{GW}}{\Delta_R} = 4\varepsilon = -2n_T$$

therefore, if GW are detected they should have an amplitude which is related to Δ_R through its spectral index n_T! Such a key check if observed would be a major triumph for the simple models of inflation (which we are assuming here).

In observable quantities, we do not actually measure Δ_R and Δ_{GW}, but only their manifestation in the spectrum of CMB fluctuations C_l (where l here means legendre multipole, 2D decomposition of the sky)

~~These~~ In terms of C_l's the relationship is more like

$$r = \frac{C_l]_{GW}}{C_l]_R} \simeq 10\,\varepsilon$$

"tensor to scalar ratio"

Classification of Inflationary models

In order to constrain inflationary model against observations it is useful to classify them according to where they land in parameter space of observables, such as (n_s-1) and r (to lowest order in slow-roll). This will gives us some idea of the type of potentials that are allowed by observational data.

Even in the context of single-field inflationary models, the number of models proposed is very large; ~~m~~however, they can be generically characterized by two mass scales in the potential: a height Λ^4 (corresponding to the energy density during inflation) and a width μ (corresponding to the change in the field value $\Delta\phi$ during inflation).

We write :
$$V(\phi) = \Lambda^4 \, f\left(\frac{\phi}{\mu}\right)$$

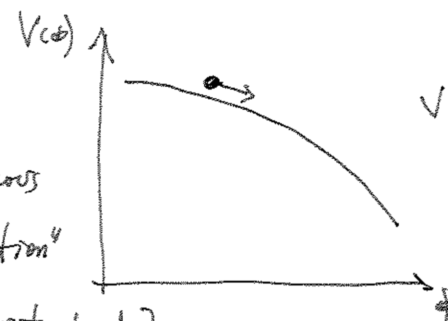
The height Λ is fixed by the normalization of density perturbations for a given model (through V/ε) ; then basically μ is the free parameter left. Different models have different f's. The relevant parameter space for distinguishing models to lowest order in slow-roll is the r-n_s plane. Since

$$\left.\begin{aligned} n_s - 1 &= 2\eta - 6\varepsilon \\ r &\simeq 10\,\varepsilon \end{aligned}\right\} \Rightarrow \quad n_s - 1 = 2\eta - \frac{3}{5} r$$

The relationship between n_s and r is through η, models can be classified in the r-n_s plane through the value of η.

i) <u>Small-field models</u> : $\eta < 0$

These are the type of potentials that arise naturally from spontaneous symmetry breaking [e.g. "new inflation" and "natural inflation", that has a cos-type potential].



$$V \sim \Lambda^4 \left[1 - \left(\frac{\phi}{\mu}\right)^p\right]$$

ϕ starts near an unstable equilibrium (defined as the origin) and rolls down the potential to a stable minimum.

Typically ε is very small in this case, e.g. for the potential above we have :

$$\begin{cases} \varepsilon \simeq \dfrac{p^2}{2} \left(\dfrac{M_{pl}}{\mu}\right)^2 \left(\dfrac{\phi}{\mu}\right)^{2p-2} \\[2ex] \eta \simeq -p(p-1) \left(\dfrac{M_{pl}}{\mu}\right)^2 \left(\dfrac{\phi}{\mu}\right)^{p-2} \end{cases}$$

Since $\phi \ll \mu$ for $p \geq 2$ ε is small. To relate $(n_s - 1)$ and r we introduce the number of e-folds N as a time parametrization of when a given mode that we see today crosses Hubble during inflation :

$$N(k) = \int_{a_*(k)}^{a_e} d\ln a = \int_{t_*}^{t_e} H\, dt = \int_{\phi_*}^{\phi_e} \frac{H}{\dot\phi}\, d\phi \underset{\substack{\uparrow \\ 3H\dot\phi \simeq -V' \\ \frac{V}{3M_{pl}^2} = H^2}}{=} \frac{1}{M_{pl}^2}\int_{\phi_e}^{\phi} \frac{V}{V'}\, d\phi$$

Now, since $\varepsilon = \frac{M_{pl}^2}{2}\left(\frac{V'}{V}\right)^2$

$$N(k) \underset{\substack{\uparrow \\ \text{take } - \text{sign} \\ \text{in } \sqrt{\varepsilon} \text{ since } V'<0}}{=} - \frac{M_{pl}^{-1}}{\sqrt{2}} \int_{\phi_e}^{\phi} \frac{d\phi}{\sqrt{\varepsilon(\phi)}} \simeq - \frac{M_{pl}}{\cancel{\sqrt{2}}} \frac{1}{M_{pl}^2} \int_{\phi_e}^{\phi} \frac{d\phi}{\cancel{\frac{p}{\sqrt{2}} \frac{M_{pl}}{\mu}} \left(\frac{\phi}{\mu}\right)^{p-1}} =$$

$$= \left(\frac{\mu}{M_{pl}}\right)^2 \frac{1}{p} \int_{\phi_e/\mu}^{\phi/\mu} (-1) \frac{dy}{y^{p-1}} \underset{\substack{\uparrow \\ p>2}}{=} \frac{-1}{p(2-p)}\, y^{2-p}\Big|_{\phi_e/\mu}^{\phi/\mu} \left(\frac{\mu}{M_{pl}}\right)^2$$

Since $p>2$ and $\phi_e/\mu > \phi/\mu$ (small field model), we have

$$N \simeq \frac{-1}{p(2-p)} \left(\frac{\phi}{\mu}\right)^{2-p} \left(\frac{\mu}{M_{pl}}\right)^2$$

then we have $n_s - 1 = 2\eta - 6\varepsilon \simeq -2p(p-1)\left(\frac{M_{pl}}{\mu}\right)^2 \left(\frac{\phi}{\mu}\right)^{p-2} - 6\varepsilon$

$$\Rightarrow \quad \boxed{n_s - 1 \simeq -\frac{2}{N}\left(\frac{p-1}{p-2}\right) - \frac{3}{8} r} \quad (p>2)$$

therefore the <u>scalar tilt (n_s-1) is negative</u>.

Note that in the above calculation we didn't need to evaluate ϕ_e (in terms of a condition for the end of inflation, $\varepsilon(\phi_e) \simeq 1$) because of the form of the potential and small-field condition the integral for N was dominated by the ϕ/μ limit, but otherwise one has to include that

For $p=2$ one recovers that,
$$r = 5(1-n_s)\, e^{-[1+N(1-n_s)]}$$

ii) <u>linear models</u> : $\eta = 0$

In this case $V(\phi) \propto \phi$ and $r = \frac{5}{3}(1-n_s)$, a line in the $n_s - r$ plane.

iii) Large-field models : $(0 < \eta \leq 2\varepsilon)$

This is typical of "chaotic inflation" scenarios where the scalar field is displaced from the minimum of the potential by an amount of a few M_{Pl} - A typical potential is

$$V(\phi) \sim \Lambda^4 \left(\frac{\phi}{\mu}\right)^p$$

Or exponential potentials $V(\phi) \sim \Lambda^4 e^{\phi/\mu}$ for which it follows that $r = 5(1-n_s)$ - For $V \sim \phi^p$ we have

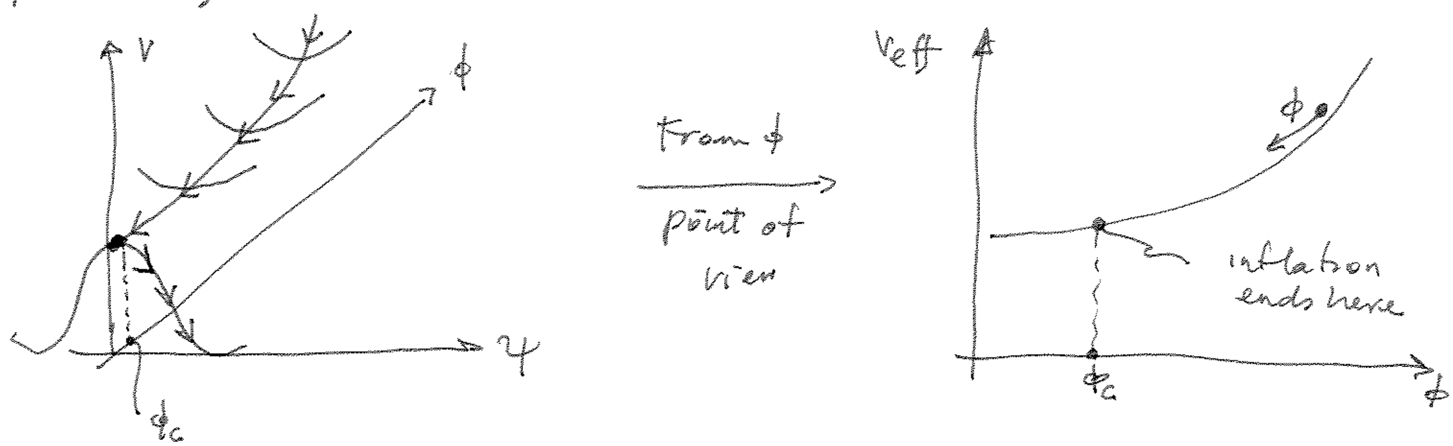
$$r = 5\left(\frac{p}{p+2}\right)(1-n_s)$$

~~check~~ for $p=4$ ~~gives~~ one can parametrize the line using N and get

$$r = \frac{10}{N+1} \qquad 1-n_s = \frac{3}{N+1}$$ again, the scalar tilt is negative

iv) hybrid models : $(0 < 2\varepsilon < \eta)$

These type of models appear frequently when trying to realize inflation in supersymmetric models - The inflaton evolves towards a minimum for large ϕ to small ϕ when an instability happens at $\phi = \phi_c$ that terminates inflation : this is due to a second field that develops a negative effective squared mass - For example:



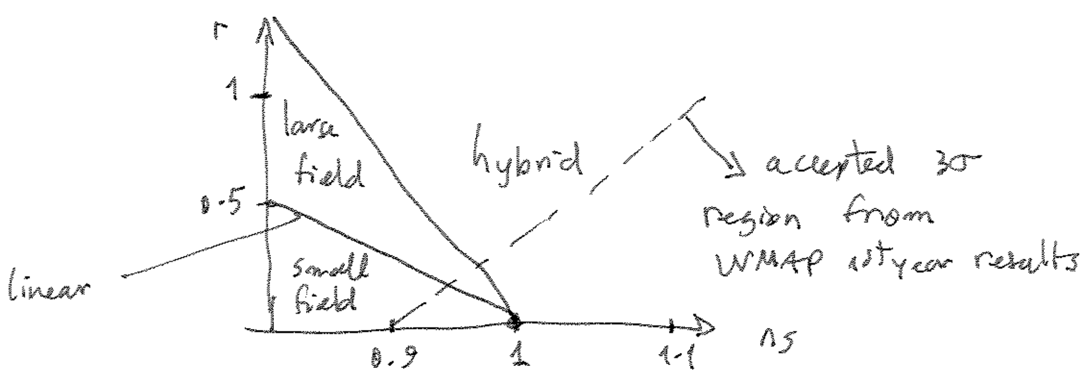
A generic potential of this type is something like

$$V \sim \Lambda^4 \left[1 + (\phi/\mu)^p\right]$$

Since the end of inflation is determined by other physics, there is a second parameter characterizing these models (note that $\varepsilon(\phi_e)=1$, e.g. is not the condition for ending inflation here). Because of this extra freedom, hybrid models fill a broad region in the $n_s - r$ plane, ~~though~~ though there is no overlap with previous models.

When $\eta > 3\varepsilon$ we can have a positive scalar tilt, this is a distinct feature of these models, although they can also have negative tilt for $2\varepsilon < \eta < 3\varepsilon$.

Summarizing, the $n_s - r$ plane looks like:



Brief summary of current constraints

The lack of detection of tensor modes from 1st year WMAP leads to a constraint on the Hubble constant during inflation for the modes we observe (let's call it H^*):

$$H^* \lessapprox 3.3 \times 10^{14}\ \text{GeV}$$

One can use this to see how WMAP + other CMB experiments rule out the inflaton potential $V(\phi) \sim \lambda \phi^4$ to 3σ. We use that for this potential

$$r \simeq \frac{10}{N+1} \qquad n_s - 1 = \frac{-3}{N+1}$$

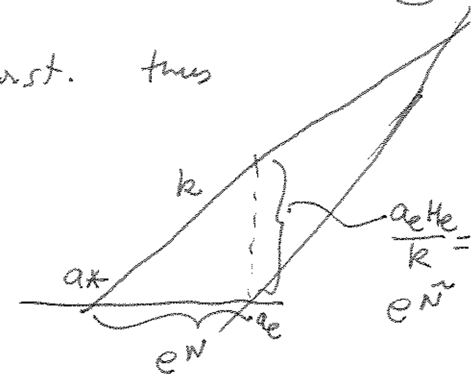
(and to next order you can also use $\frac{dn_s}{d\ln k} \simeq \frac{-3}{N(N+1)}$)

obtained by same arguments as before (using $\varepsilon(\phi_e)=1$ as defining end of inflation)

There is a bound on $N_{(k)}$ that can be obtained as follows.

For simplicity we assume expansion is close to $H \simeq$ const. thus

$$e^{\tilde{N}} = \frac{a_e H_e}{k} = a_* e^N \frac{H_e}{k} \simeq \underbrace{\frac{a_* H_*}{k}}_{1} e^N \simeq e^N$$



Now we can write H_e as an upper bound

$$H_e \lesssim H_{EQ} \left(\frac{a_{EQ}}{a_e}\right)^2 \sim \frac{H_0}{a_e^2} \sqrt{\Omega_R^0}$$

assume H does not fall faster than a^{-4} after inflation

$$H_{EQ}^2 = H_0^2 \left(\underbrace{\frac{\Omega_R^0}{a_{EQ}^4} + \frac{\Omega_M^0}{a_{EQ}^3}}_{\text{EQUAL @ } a_{EQ}} + \Omega_\Lambda^0 \right) = 2 H_0^2 \frac{\Omega_R^0}{a_{EQ}^4}$$

negligible @ a_{EQ}

where $\Omega_R^0 = 4.2 \times 10^{-5} h^{-2}$ is the radiation Ω today : $\Omega_R^0 = \left.\frac{\rho_{RAD}}{\rho_{crit}}\right|_{today}$.

$$\Rightarrow \quad a_e \lesssim \sqrt{\frac{H_0}{H_e}} \, \Omega_R^{0\,1/4} \quad \Rightarrow \quad e^{\tilde{N}} = \frac{a_e H_e}{k} \lesssim \sqrt{\frac{H_e}{H_0}} \, \frac{H_0}{k} \times 0.08 h^{-1/2}$$

Now using that $H_e \leq H^*$ (since Hubble decreases during inflation, slightly)

$$e^{\tilde{N}} \lesssim e^{60.9} \left(\frac{H_*}{10^{15}\,\text{GeV}}\right)^{1/2} \left(\frac{0.002\ \text{Mpc}^{-1}}{k}\right) \lesssim e^{60} \left(\frac{0.002\ \text{Mpc}^{-1}}{k}\right)$$

$H^* < 3.3 \times 10^{14}$ GeV

which for the k's we observe says N cannot be larger than $\simeq 62$, this for $\lambda \phi^4$ implies

$r \gtrsim 0.158$
$n_s \lesssim 0.95$

which is outside the 3σ boundary exclusion from CMB experiments [show plot].

Basically, what's going on is the following. If we don't see tensor modes then we put an upper bound on H^* or a lower bound on H^{*-1}, that means that as experiments get better H^{*-1} is pushed higher and higher. That moves a k-mode we observe to have crossed H_*^{-1} closer and closer to the end of inflation, (10)

so ε is getting larger (this is not true if : hybrid models) ~~[illegible]~~ , but if this is so, for a given scalar amplitude that we observe, since this goes as $\frac{V}{\varepsilon} \sim \frac{H_*^2}{\varepsilon}$ means lowering H_* one is lowering ε, then the contradiction (depends what shape exactly of V we have to reach this contradiction) – Pictorially,