IV) ~~Linear evolution of fluctuations~~ ~~(relativistic case)~~

Recall that in Newtonian case we had,

$$\dot{} \equiv \frac{\partial}{\partial t}\,\delta\dot\rho = -3H\,\delta\rho - 3\,\bar\rho\,\delta H \qquad \text{(Continuity)}$$

$$\delta\dot H = -2H\,\delta H - \frac{4\pi G}{3}\,\delta\rho - \frac{1}{3}\,\frac{\nabla_r^2\,\delta p}{\bar\rho} \qquad \text{(Euler)}$$

here $\delta H \equiv \frac{\vec\nabla_r\cdot\vec v}{3} = \frac{1}{a}\frac{1}{3}\vec\nabla\cdot\vec v \equiv \frac{\theta}{3a}$

Before we assumed MAT dominated universe. Let's assume for generality that

$$\bar\rho \propto a^{-3(1+w)} \qquad w = \begin{cases} \frac{1}{3} & \text{RAD} \\ 0 & \text{MAT} \\ (-1 & \text{VACUUM}) \end{cases}$$

where $w \equiv \bar P/\bar\rho$ defines equation of state

$$\Rightarrow \quad \dot{\bar\rho} = -3(1+w)\,H\bar\rho$$

In relativistic case the equations of motion are the same (in the so-called "Newtonian" conformal gauge) if we just replace $\bar\rho \longrightarrow \bar\rho + \bar P$ ~~and~~ the Newtonian equations! Then,

(From $\nabla_\mu T^\mu{}_\nu = 0$)

$$\begin{cases} \delta\dot\rho = -3H\,\delta\rho - 3(\bar\rho+\bar P)\,\delta H \\ \delta\dot H = -2H\,\delta H - \frac{4\pi G}{3}\,\delta\rho - \frac{1}{3}\,\frac{\nabla_r^2\,\delta p}{\bar\rho+\bar P} \end{cases}$$

[This is a lame way of doing this, sorry we don't have time for deriving this from GR, see 4.3. in Padmanabham]

with this we can discuss $k/aH < 1$ solutions as well.

Recall $\delta\rho = \bar\rho\,\delta$

$\Rightarrow$ Continuity reads $\cancel{-3(1+w)H\bar\rho\,\delta} + \bar\rho\,\frac{\partial\delta}{\partial t} = \cancel{-3H\bar\rho\,\delta} - \frac{(\bar\rho+\bar P)}{a}\,\theta$

$$\Rightarrow \quad \boxed{\frac{\partial\delta}{\partial\tau} = -(1+w)\,\theta + 3w\mathcal{H}\,\delta}$$

$a\,d\tau = dt$

$H = \mathcal{H}$

uler equation is,

$$-\frac{H}{a}\theta + \frac{1}{a}\dot{\theta} = -\frac{2H}{a}\theta - 4\pi G\bar{\rho}\,\delta - \frac{1}{a^2}\frac{\nabla^2\delta p}{\bar{\rho}+\bar{p}}$$

$$\Rightarrow \boxed{\frac{\partial \theta_k}{\partial \tau} + \mathcal{H}\,\theta_k = -\frac{3}{2}\mathcal{H}^2\delta_k + \frac{k^2 c_s^2}{1+w}\delta_k}$$

we for simplicity we assume $\Omega \equiv 1$.

Now, let's find 2nd order diff equation for δ_k:

$$\frac{\partial^2\delta}{\partial\tau^2} = -w'\theta - (1+w)\left[-\mathcal{H}\theta - \frac{3}{2}\mathcal{H}^2\delta + \frac{k^2c_s^2}{1+w}\delta\right] + 3w'\mathcal{H}\delta + 3w\mathcal{H}'\delta + 3w\mathcal{H}\frac{\partial\delta}{\partial\tau}$$

where $' \equiv \frac{\partial}{\partial\tau}$. Recall that,

$$\begin{cases} \mathcal{H}' = -\frac{3}{2}\mathcal{H}^2(1+w) + \mathcal{H}^2 & \text{(Friedman equation)} \\ w' = 3\mathcal{H}(w-c_s^2)(1+w) & \text{(From } \bar{p} = w\bar{\rho} \text{ and } \dot{\bar{p}} = c_s^2\dot{\bar{\rho}}) \end{cases}$$

Then, we have:

$$\frac{\partial^2\delta_k}{\partial\tau^2} = -3\mathcal{H}(w-c_s^2)\frac{\cancel{(1+w)}}{\cancel{(1+w)}}\left[3w\mathcal{H}\delta - \frac{\partial\delta}{\partial\tau}\right] + \frac{\cancel{(1+w)}}{\cancel{(1+w)}}\mathcal{H}\left[3w\mathcal{H}\delta - \frac{\partial\delta}{\partial\tau}\right]$$
$$+ (1+w)\frac{3}{2}\mathcal{H}^2\delta - k^2c_s^2\delta + 9\mathcal{H}^2(w-c_s^2)(1+w)\delta +$$
$$+ 3w\mathcal{H}^2\left[1 - \frac{3}{2}(1+w)\right]\delta + 3w\mathcal{H}\frac{\partial\delta}{\partial\tau}$$
$$= \mathcal{H}^2\delta\,[\underbrace{6w + \tfrac{3}{2}(1+w)(1-3w)}_{3w+\frac{3}{2}-\frac{9}{2}w^2}] - k^2c_s^2\delta + \mathcal{H}\frac{\partial\delta}{\partial\tau}\left[6w - 1 - 3c_s^2\right]$$

Since we are assuming $\Omega=1$, solutions will depend on time only through $\underline{a}$, so it is convenient to use $\underline{a}$ as a time ariable:

$$\frac{1}{\mathcal{H}}\frac{\partial}{\partial\tau} = \frac{\partial}{\partial \ln a} \qquad 1-\frac{3}{2}(1+w)$$

$$\frac{1}{\mathcal{H}^2}\frac{\partial^2}{\partial\tau^2} = \frac{\partial^2}{\partial(\ln a)^2} + \frac{1}{\mathcal{H}^2}\frac{\partial\mathcal{H}}{\partial\tau}\frac{\partial}{\partial \ln a}$$

Then:

$$\frac{\partial^2\delta}{\partial \ln a^2} + \left[1-\frac{3}{2}(1+w)\right]\frac{\partial\delta}{\partial \ln a} = (6w-1-3c_s^2)\frac{\partial\delta}{\partial \ln a} - \frac{k^2c_s^2}{\mathcal{H}^2}\delta$$

$$+\ \delta\left[3w+\frac{3}{2}-\frac{9}{2}w^2\right]$$

$\Rightarrow$

$$\boxed{\frac{\partial^2\delta}{\partial \ln a^2} + \left[\frac{1}{2}-\frac{15}{2}w+3c_s^2\right]\frac{\partial\delta}{\partial \ln a} + \left[\frac{9}{2}w^2-3w-\frac{3}{2}\right]\delta + \frac{k^2c_s^2}{\mathcal{H}^2}\delta = 0}$$

For matter: $c_s^2 = w = 0$, and $\frac{k^2}{\mathcal{H}^2} \ll 1$ [either $c_s = 0$ for DARK MATTER or at scales larger than Jeans]

$$\Rightarrow \frac{\partial^2\delta}{\partial \ln a^2} + \frac{1}{2}\frac{\partial\delta}{\partial \ln a} - \frac{3}{2}\delta = 0$$

try a^p $\Rightarrow$ $p^2+\frac{p}{2}-\frac{3}{2}=0 \Rightarrow \boxed{p = 1, -3/2}$

The usual growing and decaying modes (also for scales larger than Hubble radius!)

- For radiation: $c_s^2 = w = 1/3$

$$\frac{1}{2}-\frac{15w}{2}+3c_s^2 = \frac{1}{2}-\frac{5}{2}+1 = -1$$

$$\frac{9}{2}w^2-3w-\frac{3}{2} = \frac{1}{2}-1-\frac{3}{2} = -2$$

Then:

$$\frac{\partial^2\delta}{\partial \ln a^2} - \frac{\partial\delta}{\partial \ln a} - 2\delta + \frac{k^2}{3H^2}\delta = 0$$

$$\Rightarrow \boxed{\frac{\partial^2\delta}{\partial \ln a^2} - \frac{\partial\delta}{\partial \ln a} = \left(2 - \frac{k^2}{3H^2}\right)\delta}$$

So if $\frac{k^2}{3H^2} \ll 2$ ~~perturbations~~ scales larger than Hubble (which is equivalent to Jeans length in RAD)

$$\Rightarrow \frac{\partial^2\delta}{\partial \ln a^2} - \frac{\partial\delta}{\partial \ln a} - 2\delta \approx 0 \Rightarrow \delta \sim a^p$$

$$\Rightarrow p^2 - p - 2 = 0 \Rightarrow \boxed{p = 2, -1}$$

So perturbations in RAD, for $\frac{k}{aH} \ll 1$ behave as

$$\delta_k \simeq A_k a^2 + B_k a^{-1}$$

However, in the opposite limit, when $\frac{k^2}{H^2}$ is large

$$\frac{\partial^2\delta}{\partial \ln a^2} - \frac{\partial\delta}{\partial \ln a} = a^2\frac{\partial^2\delta}{\partial a^2} \simeq -\frac{k^2}{3H^2}\delta$$

$$\Rightarrow \boxed{\frac{\partial^2\delta}{\partial a^2} = -\frac{k^2}{3H^2a^4}\delta}$$

Now, in RAD $H^2 \sim \rho_R \sim a^{-4} \Rightarrow H^2a^4 = \text{const.}$

$$\Rightarrow \boxed{\delta \propto \exp\left[\pm i\frac{k\,a}{\sqrt{3}\,Ha^2}\right]} \qquad \frac{k}{aH} \gg 1$$

again, sound waves.

Evolution of CDM during radiation era

We now consider the evolution of fluctuations in dark matter component when it becomes smaller than Hubble radius during RAD. At scales $\lambda \ll H^{-1}$ Radiation is smooth (recall that Jeans length for RAD is $\sim H^{-1}$); but during RAD era ρ is dominated by $\bar{\rho}_R$, so it dominates the evolution of Hubble constant.

We have:

Newtonian evolution ($\lambda \ll H^{-1}$), but gravity due to 2 components

$$\frac{\partial^2 \delta}{\partial \tau} + \mathcal{H} \frac{\partial \delta}{\partial \tau} = 4\pi G a^2 \, \delta\rho_{TOT} = 4\pi G a^2 \left[\bar{\rho}_M \delta + \bar{\rho}_R \delta_R \right]$$

$$\simeq 4\pi G a^2 \bar{\rho}_M \, \delta$$

But now Friedmann equation is,

$$\mathcal{H}^2 = \frac{8\pi G a^2}{3} \left(\bar{\rho}_R + \bar{\rho}_M \right)$$

It is convenient to work with time variable $x = a/a_{EQ}$

$$\Rightarrow \quad x\, \bar{\rho}_R = \bar{\rho}_M \qquad \left(\text{recall } \bar{\rho}_R \sim a^{-4}, \ \bar{\rho}_M \sim a^{-3} \right)$$

$$dx = \frac{da}{a_{EQ}} = \frac{a}{a_{EQ}} \, d\ln a = \frac{a}{a_{EQ}} \frac{d\ln a}{d\tau} \, d\tau = x \mathcal{H} \, d\tau$$

$$\Rightarrow \quad \mathcal{H} \frac{\partial \delta}{\partial \tau} = \mathcal{H}^2 x \frac{\partial \delta}{\partial x}$$

$$\mathcal{H}^2 = \frac{8\pi G a^2}{3} \bar{\rho}_M \left(1 + \frac{1}{x} \right) \quad \Rightarrow \quad 4\pi G a^2 \bar{\rho}_M = \frac{3}{2} \mathcal{H}^2 \frac{x}{1+x}$$

$$\frac{\partial^2 \delta}{\partial \tau^2} = \mathcal{H} x \frac{\partial}{\partial x} \left(\mathcal{H} x \frac{\partial \delta}{\partial x} \right) = \mathcal{H}^2 x^2 \frac{\partial^2 \delta}{\partial x^2} + \mathcal{H}^2 x \frac{\partial \delta}{\partial x} + \mathcal{H} x^2 \frac{\partial \mathcal{H}}{\partial x} \frac{\partial \delta}{\partial x}$$

We need to get $\frac{\partial \mathcal{H}}{\partial x}$, From Friedmann equation get

$$2\mathcal{H}\, d\mathcal{H} = dx \, \frac{8\pi G a^2 \bar{\rho}_M}{3} \left[\frac{2}{x}\left(1+\frac{1}{x}\right) - \frac{3}{x}\left(1+\frac{1}{x}\right) - \frac{1}{x^2} \right]$$

$$\frac{d\bar{\rho}_M}{dx} = -3\frac{\bar{\rho}_M}{x} \qquad\qquad = \frac{\mathcal{H}^2}{1+1/x}$$

$$\Rightarrow \quad \frac{\partial \mathcal{H}}{\partial x} = \frac{\mathcal{H}^2}{1+\frac{1}{x}} \, \frac{1}{2\mathcal{H}} \left[-\frac{1}{x}\left(1+\frac{1}{x}\right) - \frac{1}{x^2} \right] = -\frac{\mathcal{H}}{2} \, \frac{x+2}{x(x+1)}$$

hen, we have:

$$\cancel{\mathcal{H}^2} x^2 \frac{\partial^2 \delta}{\partial x^2} + 2\cancel{\mathcal{H}^2} x \frac{\partial \delta}{\partial x} - \frac{\cancel{\mathcal{H}^2}}{2} \frac{x^2 (x+2)}{\cancel{x}(x+1)} \frac{\partial \delta}{\partial x} = \frac{3}{2} \cancel{\mathcal{H}^2} \frac{x}{1+x} \delta$$

$$2x - \frac{x(x+2)}{2(x+1)} = \frac{x(3x+2)}{2(x+1)}$$

$$\Rightarrow \quad \boxed{2x(1+x) \frac{\partial^2 \delta}{\partial x^2} + (3x+2) \frac{\partial \delta}{\partial x} = 3\delta} \qquad\qquad x \equiv a/a_{eq}$$

he solution can be guessed by setting $\frac{\partial^2 \delta}{\partial x^2} = 0 \Rightarrow \delta \propto 3x+2$, or

$$\delta_k^{+} = A_k \, (3x+2)$$

e can see that this becomes the usual growing mode in MAT era, for $x \gg 1$ $\delta \propto a$. In the RAD era, however, $x \ll 1$ and growth is suppressed, this is because extra contribution to $\mathcal{H}$ speeds up the expansion of the universe leading to slower growth.

Thus, even though $\lambda > \lambda_J$ the dark matter perturbations do not grow significantly during RAD era (for scales $\lambda < H^{-1}$).

Remember for $\lambda > H^{-1}$, $\delta \propto a^2$ during RAD era. To do the matching @ $\lambda = H^{-1}$, we need to ~~develo~~ the decaying mode.

This is found by taking $\delta = (3x+2)\,f$ and solving for f: 7

$$\delta_{\vec k}^{-} = B_{\vec k}\left[(3x+2)\ln\left(\frac{\sqrt{1+x}+1}{\sqrt{1+x}-1}\right) - 6\sqrt{1+x}\right]$$

which gives:

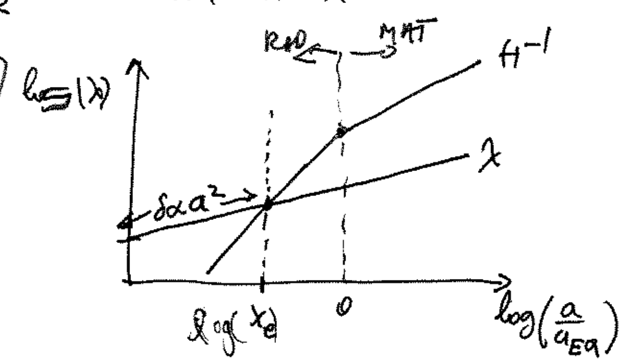
$$\delta_{\vec k}^{-} = \begin{cases} B_{\vec k}\left(2\ln\frac{4}{x} - 6\right) & x \ll 1 \\ \frac{8}{15}B_{\vec k}\,\frac{1}{x^{3/2}} & x \gg 1 \end{cases} \qquad (*)$$

he latter is what we expect, $\delta \propto a^{-3/2}$ during MAT era.

owmuch does a mode that enters well in the RAD era ($x_{enter} \equiv x_e \ll 1$) row until $x = 1$? Since the mode has been super Hubble for a long ime, we assume that by the time it enters is in the growing mode for $\lambda > H^{-1}$, i.e.

$$\delta(x) \equiv x^2 \qquad \text{for } x < x_e \quad \text{when } \lambda > H^{-1}$$

[his is the same as for RAD perturbations, we are here assuming adiabatic (or curvature) perturbations, more on this later]



owever, this mode will not correspond merely to a growing mode $(3x+2)$ by the time it becomes sub-Hubble, it will be a linear combination of growing and decaying modes. To find that combination we impose matching conditions.

since $x_e \ll 1$ we can use the approx $(*)$

At $x = x_e$:

$$\begin{cases} x_e^2 = 2A - 6B + 2B\ln\frac{4}{x_e} & \left(\delta^{outside} = \delta^{inside}\right) \\ 2x_e = 3A - 2B\,\frac{1}{x_e} & \left(\frac{\partial\delta}{\partial x}^{outside} = \frac{\partial\delta}{\partial x}^{inside}\right) \end{cases}$$

$$\Rightarrow \quad B \approx -x_e^2$$

en, $x_e^2 - 6x_e^2 = 2A - 2x_e^2 \ln(4/x_e)$

$$\Rightarrow \quad A = \frac{x_e^2}{2}\left[2\ln\left(\frac{4}{x_e}\right) - 5\right]$$

(A is $\mathcal{O}(x_e^2)$ so throwing away 3A was OK to solve for B)

, once it crosses we have conversion into growing and decaying modes:

$$\delta(x) = x_e^2\left[\ln\left(\frac{4}{x_e}\right) - \frac{5}{2}\right](3x+2) - x_e^2\left[(3x+2)\ln\frac{\sqrt{1+x}+1}{\sqrt{1+x}-1} - 6\sqrt{1+x}\right]$$

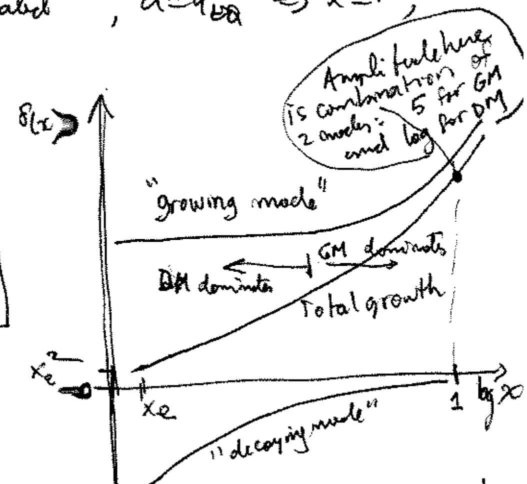
$$\underset{x_e \ll 1}{\approx} \quad x_e^2 \ln(x_e^{-1})\,(3x+2)$$

Note the negative sign!

o, by the time the universe becomes MAT dominated, $a = a_{EQ} \Rightarrow x = 1$,

e have an amplification

$$\boxed{\frac{\delta(x=1)}{\delta(x_e)} = \frac{5\cancel{x_e^2}\ln(x_e^{-1})}{\cancel{x_e^2}} = 5\ln\left(\frac{a_{EQ}}{a_{enter}}\right)}$$



hich is only logarithmic! (This only holds for $x_e \ll 1$, as $x_e \to 1$ growth is closer to x)

Note that ~~[illegible]~~ the amplification factor depends on the wavelength f the mode (through a_{enter}), so different modes will be amplified slightly differently, leading to a change in the power spectrum.

Summarizing the evolution of perturbations: (radiation, Dark Matter, baryons)

(these we did not show, but can be shown to hold for adiabatic perturb.)

	Epoch	δ_R	δ_{DM}	δ_B
$\frac{k}{aH} < 1$	$\lambda > H^{-1}$, RAD era	a^2	a^2	a^2
$\frac{k}{aH} > 1$	$\lambda < H^{-1}$, RAD era	oscillates	$\ln a$	oscillates
"	$\lambda < H^{-1}$, before DEC	"	a	"
"	$\lambda < H^{-1}$, after DEC	"	a	a (For $\lambda > \lambda^B_{Jeans}$)

γ-pressure

Coupled to γ's

only well into RAD era, Universe expands too fast and RAD is smooth on those scales

i.e. DM is not domina