XI The curvature perturbation: relation to Φ

Recall Homogeneous Friedmann equation:

$$H^2 = \frac{8\pi G}{3}\bar{\rho} - \frac{K}{a^2}$$

(we take K instead of k, not to confuse with Fourier modes!)

the 3D curvature (Ricci) is $\quad {}^3R = \frac{6K}{a^2}$

$$\Rightarrow H^2_{(t)} = \frac{8\pi G}{3}\bar{\rho} - \frac{1}{6}\,{}^3R$$

For the perturbed case, we define the curvature perturbation in analogous way:

(we assume $K=0$)

$$H^2(\bar{x},t) = \frac{8\pi G}{3}\rho(\bar{x},t) + \underbrace{\frac{2}{3a^2}\nabla^2\mathcal{R}(\bar{x},t)}_{\equiv -\delta K/a^2}$$

this corresponds in Fourier space to $\quad -\frac{2}{3}\frac{k^2}{a^2}\mathcal{R}_{\bar{k}} = -\frac{1}{6}\Big(\underbrace{4\frac{k^2}{a^2}\mathcal{R}_{\bar{k}}}_{{}^3R}\Big)$ $\quad {}^3R$: spatial curvature for (scalar) perturbations

In what follows we assume $\underline{K=0}$.

We want to find how $\mathcal{R}_{\bar{k}}$ evolves outside Hubble radius, to connect perturbations generated by inflation to how they come back inside Hubble radius later. Also, we seek the connection of $\mathcal{R}_{\bar{k}}$ to $\Phi_{\bar{k}}$, the gravitational potential so we can translate fluctuations in $\mathcal{R}$ to density fluctuations. [Remember that $\mathcal{R}_{\bar{k}} = -\left[\frac{H}{\dot{\phi}}\delta\phi_{\bar{k}}\right]_{t_*}$ from inflation]

To find evolution of $\mathcal{R}_{\bar{k}}$, we recall the evolution of fluctuations in relativistic case,

$$\begin{cases} \dot{\delta\rho} = -3H\,\delta\rho - 3(\bar{\rho}+\bar{p})\,\delta H \\ \dot{\delta H} = -2H\,\delta H - \frac{4\pi G}{3}\delta\rho - \frac{1}{3}\frac{\nabla^2\delta p}{a^2(\bar{\rho}+\bar{p})} \end{cases}$$

where we are using

$$\begin{cases} H(\bar{x},t) \equiv H(t) + \delta H(\bar{x},t) \\ \delta\rho(\bar{x},t) = \bar{\rho}(t)\,\delta(\bar{x},t) \end{cases}$$

To find evolution of R_k we take time derivative from

$$H^2_{(\vec{x},t)} = \frac{8\pi G}{3}\rho(\vec{x},t) + \frac{2}{3a^2}\nabla^2 \mathcal{R}(\vec{x},t)$$

e homogeneous part will not involve $\mathcal{R}$, obviously, so it will : an identity involving $H(t)$, $\dot H(t)$, $\bar\rho$, $\dot{\bar\rho}$, the usual acceleration equation. , we only need to look at time derivative of above in first order perturbations:

$$2H\,\delta\dot H_k + 2\,\delta H_k\,\dot H = \frac{8\pi G}{3}\delta\dot\rho - \frac{2}{3}\frac{k^2}{a^2}\dot R_k + \frac{4k^2}{3a^2}R_k H$$

$$\Rightarrow \quad \dot R_k = \frac{3a^2}{2k^2}\left[\frac{8\pi G}{3}\delta\dot\rho - 2\left(H\,\delta\dot H_k + \delta H_k\,\dot H\right)\right] + 2R_k H$$

(linear combination of homogeneous Friedman)

Now, we use equations of motion for $\delta\dot\rho$, $\delta\dot H$ plus $\dot H = -4\pi G(\bar\rho + \bar p)$

Thus,

$$\dot R_k = \frac{3a^2}{2k^2}\left\{\frac{8\pi G}{3}\left[-3H\,\delta\rho - 3(\bar\rho+\bar p)\,\delta H\right] - 2H\left[-2H\,\delta H - \frac{4\pi G}{3}\delta\rho - \frac{1}{3}\frac{\nabla^2\delta p}{a^2(\bar\rho+\bar p)}\right] - 2\,\delta H\left[-4\pi G(\bar\rho+\bar p)\right]\right\} + 2R_k H$$

$$= \frac{3a^2}{2k^2}\left\{\delta\rho\left[\frac{8\pi G}{3}(-3H) + \frac{8\pi G}{3}H\right] + \delta H\left[-\cancel{8\pi G(\bar\rho+\bar p)} + 4H^2 + \cancel{8\pi G(\bar\rho+\bar p)}\right]\right\}$$

$$+ \cancel{\frac{3a^2}{2k^2}}\,\cancel{\frac{k^2}{3a^2}}\,\frac{\delta p_k}{\bar\rho+\bar p}(-\cancel{2}H) + 2R_k H$$

$$\Rightarrow \quad \dot R_k = -H\frac{\delta p_k}{\bar\rho+\bar p} + \frac{3a^2}{2k^2}\left[\frac{4k^2}{3a^2}R_k H + 4H^2\,\delta H - 2H\,\frac{8\pi G}{3}\delta\rho\right]$$

ut from perturbing local Friedmann we have:

$$2H\,\delta H = \frac{8\pi G}{3}\delta\rho - \frac{2k^2}{3a^2}R_k \quad \overset{\times 2H}{\Rightarrow} \quad \frac{4k^2}{3a^2}R_k H + 4H^2\,\delta H - 2H\,\frac{8\pi G}{3}\delta\rho = 0$$

Therefore,
$$\dot{R}_k = -H \frac{\delta p_k}{\bar\rho + \bar p}$$

see how much R_k changes in a Hubble time, we look at,

$$\frac{\dot R_k}{H R_k} = \frac{1}{H}\frac{\partial \ln R_k}{\partial t} = -\frac{\delta p_k}{R_k}\frac{1}{\bar\rho+\bar p} \underset{\text{Poisson}}{=} \frac{2}{3}\frac{\delta p_k}{\delta \rho_k}\frac{k^2}{\mathcal{H}^2}\frac{\Phi_k \bar\rho}{\bar\rho+\bar p}\frac{1}{R_k}$$

$$\left(-k^2\Phi_k = \frac{3}{2}\mathcal{H}^2 \frac{\delta\rho_k}{\bar\rho}\right)$$

[technically, since $\mathcal{R}$ is gauge inv. we can evaluate $\mathcal{R}$ in any gauge, Poisson as written is true in total matter gauge]

$$\Rightarrow \boxed{\frac{1}{H}\frac{\partial \ln R_k}{\partial t} = \frac{2}{3}\left(\frac{\delta p_k}{\delta\rho_k}\right)\left(\frac{k}{aH}\right)^2 \frac{\Phi_k}{1+w}\frac{1}{R_k}}$$

So, for $\frac{1}{H}\frac{\partial \ln R_k}{\partial t}$ to be negligible outside the Hubble radius, $\left(\frac{k}{aH}\right) \ll 1$

all we need is to show that $\frac{\delta p_k}{\delta\rho_k}$ and $\frac{\Phi_k}{(1+w)R_k}$ do not change significantly with k or at least are of order unity.

For adiabatic fluctuations (curvature fluctuations), which we are considering here, $\frac{\delta p_k}{\delta \rho_k} = \frac{\dot{\bar P}}{\dot{\bar\rho}} = c_s^2$ which is of order unity and does not depend ~~on~~ k significantly for super-Hubble evolution during times of interest.

Adiabatic fluctuations are so called because since they correspond to fluctuations in the local value of the local curvature, all species participate equally (by equivalence pple.),

$$\frac{\delta n_\gamma}{n_\gamma} = \frac{\delta n_B}{n_B} = \frac{\delta n_{DM}}{n_{DM}} = \frac{\delta s}{s} \quad (s: \text{entropy density})$$

recall $s \propto a^{-3}$

$$\Rightarrow \delta(n_i/s) = \frac{\delta n_i}{s} - \frac{n_i}{s^2}\delta s = \frac{n_i}{s}\left(\frac{\delta n_i}{n_i} - \frac{\delta s}{s}\right) = 0$$

$\Rightarrow$ the fluctuation in the number per comoving volume vanishes $\Rightarrow$ adiabatic
(Also, for same reason $\delta(S/n) = 0$, entropy per matter particle is conserved)

That is, as we expand or contract some volume element by some factor all species experience the same change in number density.

There are also <u>isocurvature fluctuations</u>, in which the total (4)
energy density perturbation vanishes, i.e. 2 components where
e.g. $\delta\rho_\gamma + \delta\rho_{com} = 0$. We shall not consider these perturbations
here (mostly because their contribution to the total energy density
must be small compared to adiabatic fluctuations, from constraints
coming from CMB fluctuations).

We now want to see that $\frac{\Phi_k}{(1+w)R_k}$ is not large. We get such a result by
looking at the continuity equation,

$$\delta\dot{\rho}_k = -3(\bar{\rho}+\bar{p})\,\delta H_k - 3H\,\delta\rho_k$$

Poisson equation $-k^2\Phi_k = \frac{3}{2}H^2\delta_k = 4\pi G a^2 \delta\rho_k$; $\delta_k \equiv \frac{\delta\rho_k}{\bar{\rho}}$

and local Friedmann equation: $2H\delta H_k = \frac{8\pi G}{3}\delta\rho_k - \frac{2k^2}{3a^2}R_k$

From Poisson,

$$\delta\dot{\rho}_k = -\frac{k^2\dot{\Phi}_k}{4\pi G a^2} + \frac{2k^2\Phi_k}{4\pi G a^2}H$$

continuity $\Rightarrow$ (Friedmann, Poisson)

$$-\frac{k^2\dot{\Phi}_k}{4\pi G a^2} + \frac{2k^2\Phi_k}{4\pi G a^2}H = -3(\rho+\bar{p})\frac{1}{2H}\Big[\underbrace{\frac{8\pi G}{3}\delta\rho_k}_{=-\frac{2}{3}\frac{k^2\Phi_k}{a^2}} - \frac{2}{3}\frac{k^2}{a^2}R_k\Big] + 3H\frac{k^2\Phi_k}{4\pi G a^2}$$

Multiply by $\frac{2}{3H}\frac{4\pi G a^2}{-k^2}$,

$$\frac{2}{3}H^{-1}\dot{\Phi}_k - \frac{4}{3}\Phi_k = \frac{-\cancel{3}\bar{\rho}(1+w)}{\cancel{2}H}\frac{\cancel{2}}{\cancel{3}H}\frac{4\pi G a^2}{-k^2}\left(-\frac{2}{3}\frac{k^2}{a^2}\right)(\Phi_k+R_k) + \cancel{\frac{3Hk^2\Phi_k}{4\pi G a^2}\frac{2}{3H}\frac{4\pi G a^2}{-k^2}}$$

$$\Rightarrow \frac{2}{3}H^{-1}\dot{\Phi}_k = \Phi_k\Big(\frac{4}{3} - (1+w)\underbrace{\frac{4\pi G\bar{\rho}}{H^2}\frac{2}{3}}_{=1} - 2\Big) + (1+w)\underbrace{\frac{4\pi G\bar{\rho}}{H^2}\frac{2}{3}}_{=1}R_k$$

$$\frac{4}{3} - 2 - (1+w) = \frac{4-6-3-3w}{3} = -\frac{5+3w}{3}$$

hen:
$$\boxed{\frac{2}{3}H^{-1}\dot{\Phi}_k + \frac{5+3w}{3}\Phi_k = -(1+w)\,R_k}$$

uring epoch when $\dot{w} \simeq 0$, the "growing mode" solution is to me $\dot{\Phi} \simeq 0$ ~~and~~ then

(i.e. the equation above has an homogeneous solution $\Phi_k \sim a^{-\frac{5+3w}{2}}$ which we ignore, only take the particular solution obtained from $\dot{\Phi} \simeq 0$)

$$\Phi_k = \frac{-3(1+w)}{5+3w} R_k \qquad (*)$$

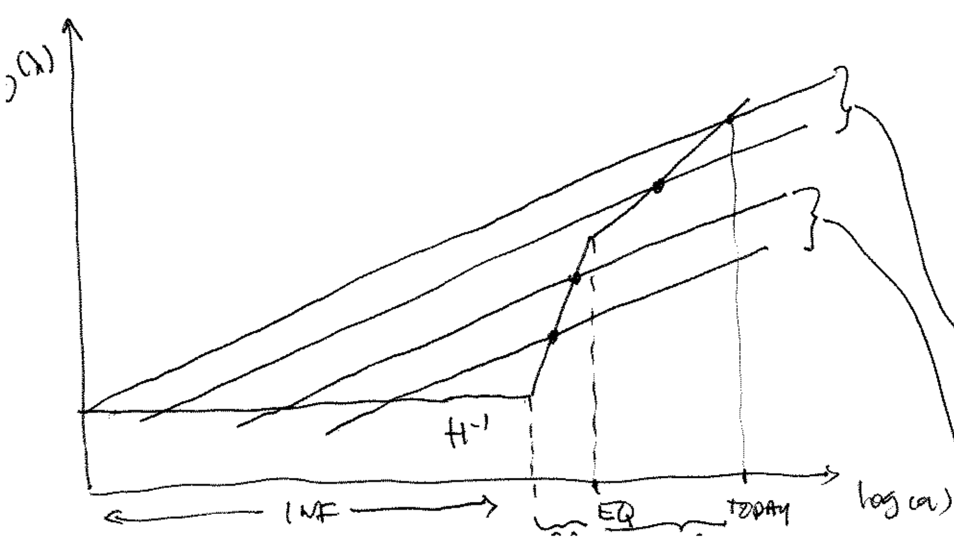
~~Ad contrast~~ Therefore $\frac{\Phi_k}{(1+w)R_k} = \frac{-3}{5+3w}$ which is independent of time

d scale and order unity – ~~[illegible]~~ Equation (*) turns out to be valid en during inflation, when $w+1 = \dot{\phi}^2/V \simeq 0 \Rightarrow \Phi_k = -\frac{3}{2}(1+w)R_k$ –

herefore as a result, when perturbations cross back in the ubble radius, they will do so as ~~[illegible]~~ a fluctuation in the ravitational potential (The constant of proportionality depends whether the modes ~~come~~ back in RAD or MAT era) –

Now we are ready to see how the power spectrum evolves since inflation until after decoupling:

The Linear Power Spectrum: Transfer Function



[Unfortunately plot is not to scale, RAD epoch should span more in log(a)]

Basic idea: for modes that enter during MAT, we see primordial perturbation, i.e. $\Phi_k \propto R_k$ when it crosses and after that $\Phi_k = $ const.

Modes that enter in RAD are suppressed

So, from discussion above, we have that $\Phi_k = -\frac{3(1+w)}{5+3w} R_k$ (6)

Hubble radius crossing. Remember that the power spectrum the curvature perturbation is Harrison-Zel'dovich (approximately, if $n_s = 1$):

$$P_R(k) \propto k^{-3} \quad \Rightarrow \quad P_\Phi(k) \propto k^{-3}$$

his is for scales that are crossing <u>now</u>.

ir scales that are a bit smaller, but still cross during the MAT era, othing happens since the gravitational potential stays constant for $\Omega_m = 1$ as we assume now) in linear perturbation theory,

$$-k^2 \Phi_k = \frac{3}{2} \mathcal{H}^2 \Omega_m \delta_k = \frac{3}{2} \mathcal{H}^2 \delta \propto \mathcal{H}^2 a \propto H^2 a^3 \propto \frac{1}{t^2} \left(t^{2/3}\right)^3 = \text{const}$$

hus, for $k \lesssim k_{eq} = \left(\frac{14}{\Omega_m h^2}\right)^{-1} \text{Mpc}^{-1}$, the comoving wavenumber that crosses Hubble radius at matter-radiation equality, we have that <u>today</u>,

$$P_\Phi(k) \propto P_R(k) \propto k^{-3 + (n_s - 1)}$$

- For modes that cross during RAD era, recall that density fluctuations grow only logarithmically (for modes that enter well before EQ) — Then after they cross Hubble radius,

$$-k^2 \Phi_k = \frac{3}{2} \mathcal{H}^2 \delta_k \propto H^2 a^2 \ln a \propto \frac{1}{\underset{a^4\ (a \propto t^{1/2})}{t^2}} a^2 \ln a \propto \frac{\ln a}{a^2}$$

Therefore these perturbations are suppressed by a factor:

$$\ln\left[\frac{a_{EQ}}{a_e(k)}\right] \left(\frac{a_e(k)}{a_{EQ}}\right)^2$$

By the time it reaches us now (after EQ, they stay constant $, \Phi_k$), where $a_e(k)$ is the scale factor at the time the mode k enters the Hubble radius.

The condition that enters is

$$\lambda_{\text{comoving}}\, a_e(k) = \frac{2\pi}{k}\, a_e(k) = H^{-1}_{[a_e(k)]} \propto t_e \propto a_e^2(k)$$

$$\Rightarrow \quad a_e(k) \propto \frac{1}{k} \qquad \Rightarrow \quad \frac{a_e(k)}{a_{EQ}} = \frac{k_{eq}}{k}$$

Then, the suppression factor is ~~then~~,

$$\ln\left(\frac{k}{k_{eq}}\right) \left(\frac{k_{eq}}{k}\right)^2 \qquad (k_{eq} < k)$$

So, for $k > k_{eq}$:

$$P_\Phi(k) \propto P_R(k) \left[\ln\left(\frac{k}{k_{eq}}\right)\left(\frac{k_{eq}}{k}\right)^2\right]^2 \qquad k > k_{eq}$$

In addition, at scales small enough where free-streaming can take place and erase fluctuations, the power spectrum will be additionally suppressed,

$$P_\Phi(k) \rightarrow P_\Phi(k) \text{ ``} e^{-(k/k_{FS})^2} \text{''}$$

where k_{FS} depends on the mass of the dark matter particle, recall that

$$l_{FS} \approx 0.5 \text{ Mpc} \quad (\Omega_{DM} h^2)^{1/3} \left(\frac{m}{1\,\text{keV}}\right)^{-4/3}$$

We now put everything together. Since the density fluctuation is related to the gravitational potential through Poisson equation, we have:

$$\delta_k = -\frac{2}{3}\left(\frac{k}{\mathcal{H}}\right)^2 \Phi_k = +\frac{2}{3}\left(\frac{k}{\mathcal{H}}\right)^2 \frac{3(1+w)}{5+3w} R_k \underset{\substack{w\approx 0 \\ \text{during MAT}}}{=} \frac{2}{5}\left(\frac{k}{\mathcal{H}}\right)^2 R_k$$

erefore, at large scales $k < k_{eq}$, the power spectrum of density perturbations s,

$$P_\delta(k) = \frac{4}{25}\left(\frac{k}{\mathcal{H}}\right)^4 P_R(k) \quad \propto k^4 \, k^{-3+(n_s-1)} \propto k^{n_s} \qquad (k \ll k_{eq})$$

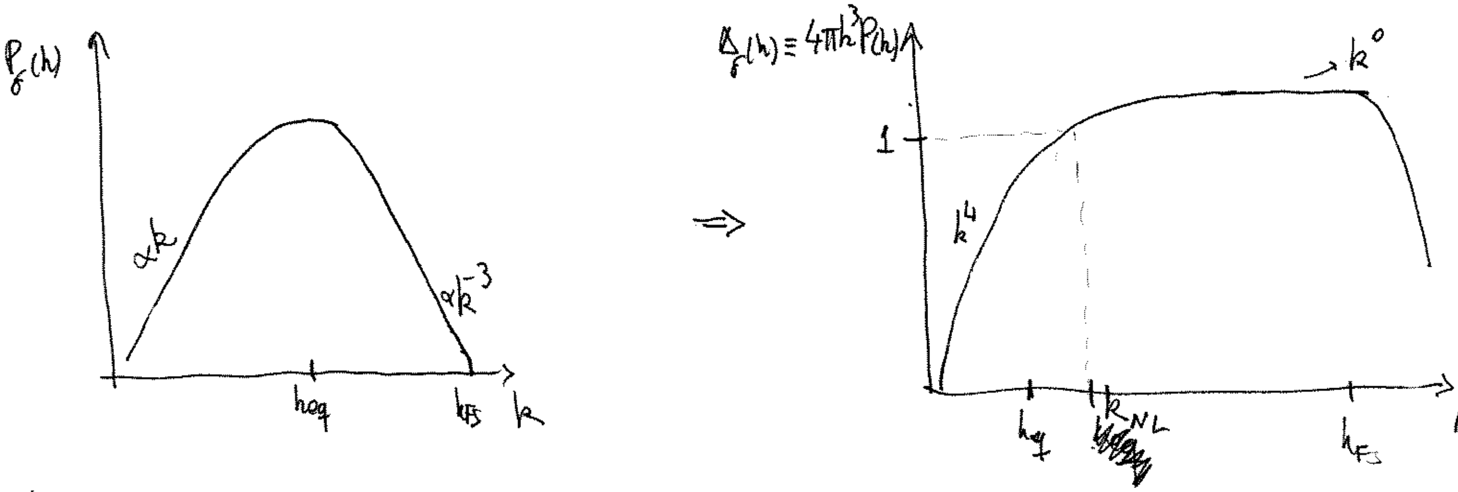
here n_s is close to unity (Harrison-Zel'dovich) for inflationary perturbations. At smaller scales this behavior is changed by the suppression of growth during RAD epoch. It is customary to define a <u>transfer function</u> $T(k)$, so that,

$$P_\delta(k) = \frac{4}{25}\left(\frac{k}{\mathcal{H}}\right)^4 P_R(k) \; T^2(k)$$

where $T(k)$ has the behavior:

$$T(k) = \begin{cases} 1 & k \ll k_{eq} \\ \ln\left(\frac{k}{k_{eq}}\right)\left(\frac{k_{eq}}{k}\right)^2 & k_{FS} \gtrsim k \gtrsim k_{eq} \\ 0 & k \gg k_{FS} \end{cases}$$

The power spectrum thus has the following shape (for HZ, $n_s = 1$)



A couple of things to note,

- At large scales we see the primordial power from inflation
- The characteristic scale k_{eq} is the comoving Hubble radius @ EQ and depends on $\Omega_m h$ [if k is in units of h/Mpc], so this peak tells us about how much dark matter there is (when $\Omega_{DM} \gg \Omega_B$)
- At scales $k \gtrsim k_{NL}$ where $\Delta_\delta(k) \gtrsim 1$ non-linear corrections to the power spectrum become important (remember we were using only linear PT so far)
- At scales $k \gtrsim k_{FS}$ power is suppressed due to free streaming (though this can be regenerated by non-linear effects)

[show current status from galaxy surveys]

I forgot (almost) to mention that a reasonable fit to the transfer function is given by the so-called BBKS form:

$$T(q) = \frac{\ln(1+2.34q)}{2.34q}\left[1+3.89q+(16.1q)^2+(5.46q)^3+(6.71q)^4\right]^{-1/4}$$

where $q = \frac{k}{\Gamma h}\,\mathrm{Mpc}^{-1}$ $\qquad \Gamma \equiv \Omega_m h \exp\left[-\Omega_B - \sqrt{2h}\,\frac{\Omega_B}{\Omega_M}\right] \underset{\Omega_B \ll \Omega_M}{\approx} \Omega_m h$

This is a numerical fit to a full numerical solution.

Note that it has the right asymptotics:

$$T(q) \sim \begin{cases} 1 & q \to 0 \\ \frac{\ln(q)}{q^2} & q \to \infty \end{cases}$$