CMB Anisotropies

The perturbations generated by inflation (or any other mechanism people may come up with in the future!) lead to fluctuations in the photon energy density, and thus on their temperature. We now discuss how this arises and what information does it provide on cosmological parameters.

Before recombination free electrons act as a glue between photons and baryons. Indeed, near recombination ($z \sim 10^3$ and assuming $\Omega_b h^2 \simeq 0.02$) the comoving mean free path of a photon is just $\lambda_\gamma \sim$ few Mpc, pretty small by cosmological standards. Therefore, on scales $\lambda > \lambda_\gamma$ photons are tightly coupled to the electrons by Thomson scattering, which in turn are tightly coupled to the baryons by Coulomb interactions.

As a result of this, the cosmological plasma was a tightly coupled photon-baryon fluid and the spectrum of anisotropies can be explained almost completely by analizing the behavior of this pre-recombination fluid. In particular the bulk velocities are defined by a single fluid velocity $v_\gamma = v_b$ and the photons carry no anisotropy in the rest frame of the baryons.

Photons are conserved, continuity gives:

$$\frac{\partial n_\gamma}{\partial \tau} + 3 n_\gamma \mathcal{H} + \vec{\nabla} \cdot [n_\gamma \cdot \vec{v}_\gamma] = 0$$

Note in the absence of perturb.

$$\frac{\partial n_\gamma}{\partial \tau} + 3 \mathcal{H} n_\gamma = 0 \Rightarrow n_\gamma \propto a^{-3}$$

as it should be.

Linearizing as usual: $n_\gamma = \bar{n}_\gamma + \delta n_\gamma$

$$\frac{\partial \delta n_\gamma}{\partial \tau} + 3 \mathcal{H}\, \delta n_\gamma + \bar{n}_\gamma \vec{\nabla} \cdot \vec{v}_\gamma = 0 \Rightarrow \frac{\partial}{\partial \tau}\left(\frac{\delta n_\gamma}{\bar{n}_\gamma}\right) + \vec{\nabla} \cdot \vec{v}_\gamma = 0$$

We define a temperature fluctuation $\frac{\delta T}{T}$ by (since $\bar{n}_\gamma \propto T^3$)

$$\frac{\delta n_\gamma}{n_\gamma} = 3 \frac{\delta T}{T} \Rightarrow \boxed{\frac{\partial}{\partial \tau}\left(\frac{\delta T}{T}\right) + \frac{1}{3} \vec{\nabla} \cdot \vec{v}_\gamma = 0}$$

To consider momentum conservation, recall that in the relativistic ② le momentum is $(\rho+p)v$ for a perfect fluid. The ratio of momentum densities for baryons to photons is:

$$R \equiv \frac{(\bar\rho_b+\bar p_b)\,v_b}{(\bar\rho_\gamma+\bar p_\gamma)\,v_\gamma} \underset{\substack{\uparrow \\ \text{tight} \\ \text{coupling}}}{=} \frac{\bar\rho_b+\bar p_b}{\bar\rho_\gamma+\bar p_\gamma} \underset{\substack{\uparrow \\ \bar p_b = 0 \\ p_\gamma = \frac{1}{3}\bar\rho_\gamma}}{=} \frac{3}{4}\frac{\bar\rho_b}{\bar\rho_\gamma} \simeq 0.6\ \frac{\Omega_b h^2}{0.02}\ \frac{1000}{1+z}$$

o while $R \ll 1$ well before REC, R becomes of order 1 @ REC. Also, the ratio of MAT to RAD is

$$\frac{\bar\rho_m}{\bar\rho_R} \simeq 3.6\ \frac{\Omega_m h^2}{0.15}\ \frac{1000}{1+z}$$

o it is a reasonable approximation to ignore radiation as a first step $\bar\rho_m/\bar\rho_R > 1$ for REC). In the following we will work in the approximation $\bar\rho \simeq \bar\rho_m$, and therefore use $\Omega_m = 1$ results, in particular that a gravitational potential Φ is constant in time. Most of the time we ill also use $\dot R = 0$ for simplicity. Relaxing these assumptions does not ead to a significant difference from the picture discussed here.

Momentum density for the baryon-photon fluid is

$$(\rho_\gamma+p_\gamma)v_\gamma + (\rho_b+p_b)v_b \underset{\substack{\uparrow \\ \text{tight} \\ \text{coupling}}}{\simeq} (\rho_\gamma+p_\gamma+\rho_b+p_b)v_\gamma = (1+R)(\rho_\gamma+p_\gamma)v_\gamma$$

Then conservation reads:

$$\frac{\partial}{\partial\tau}\left[(1+R)(\rho_\gamma+p_\gamma)\vec v_\gamma\right] + 4\mathcal{H}\left[(1+R)(\rho_\gamma+p_\gamma)\right]\vec v_\gamma = -\vec\nabla p_\gamma - (1+R)(\rho_\gamma+p_\gamma)\vec\nabla\Phi$$

Since velocities are first order, linearizing we just replace $(\rho_\gamma+p_\gamma) \to (\bar\rho_\gamma+\bar p_\gamma)$

$$\boxed{\frac{\partial \vec v_\gamma}{\partial\tau} + \frac{R'}{1+R}\vec v_\gamma = -\frac{\vec\nabla p_\gamma}{(1+R)(\bar\rho_\gamma+\bar p_\gamma)} - \vec\nabla\Phi} \qquad {}' \equiv \frac{\partial}{\partial\tau}$$

Now we combine the two conservation laws as usual to derive an evolution

· equation for $\frac{\delta T}{T}$:

$$\frac{\partial^2}{\partial \tau^2}\left(\frac{\delta T}{T}\right) + \frac{1}{3}\underbrace{\frac{\partial}{\partial \tau}\left(\bar{\nabla}\cdot\bar{v}_\gamma\right) = 0}_{-\frac{R'}{1+R}\bar{\nabla}\cdot\bar{v}_\gamma - \frac{\nabla^2 p_\gamma}{(1+R)(\bar{\rho}_\gamma+\bar{p}_\gamma)} - \nabla^2\Phi}$$

itch to Fourier space , $\nabla^2 \to -k^2$ and using $\partial/\partial\tau = \,'$

$$\left(\frac{\delta T}{T}\right)_k'' + \frac{R'}{1+R}\left(\frac{\delta T}{T}\right)' + \frac{k^2}{3}\frac{p_k}{(1+R)(\bar{\rho}_\gamma+\bar{p}_\gamma)} = -\frac{k^2}{3}\Phi_k$$

t since $\underset{\substack{\uparrow \\ c_s^2=1/3 \\ \text{for RAD}}}{p_k} = \frac{1}{3}\delta\rho_k = \frac{4}{3}\left(\frac{\delta T}{T}\right)_k \bar{\rho}_\gamma$ then,

$$\frac{p_k}{3}\frac{1}{(1+R)(\bar{\rho}_\gamma+\bar{p}_\gamma)} = \frac{1}{3}\left(\frac{\delta T}{T}\right)_k \frac{\cancel{4/3\,\bar{\rho}_\gamma}}{(1+R)\,\cancel{4/3\,\bar{\rho}_\gamma}} \equiv c_s^2\left(\frac{\delta T}{T}\right)_k$$

ere $c_s^2 = \frac{1}{3}\frac{1}{1+R}$ is the effective sound speed of the baryon-photon fluid.

en baryons become important c_s^2 is reduced as baryons add extra mass but significant pressure. In the limit $R \to 0$, $c_s^2 \to 1/3$.

Then we obtain

$$\boxed{\left(\frac{\delta T}{T}\right)_k'' + \frac{R'}{1+R}\left(\frac{\delta T}{T}\right)_k' + c_s^2k^2\left(\frac{\delta T}{T}\right)_k = -\frac{k^2}{3}\Phi_k}$$

is is the equation of motion for a damped & forced harmonic oscillator!

et's discuss this equation, starting from the simplest case, $R=R'=0$ and $\Phi_k=0$, i.e. no damping and no forcing (ignoring baryons). Then

$$\left(\frac{\delta T}{T}\right)_k'' + c_s^2k^2\left(\frac{\delta T}{T}\right)_k \simeq 0$$

Then the general solution is

$$\left(\frac{\delta T}{T}\right)_k(\tau) = \left(\frac{\delta T}{T}\right)_k(0)\cos(ks) + \left(\frac{\delta T}{T}\right)_k'(0)\,\frac{\sin(ks)}{kc_s}$$

where the $\underset{\text{(comoving)}}{\text{sound horizon}}$ is defined as $s \equiv \int_0^\tau c_s\,d\tau$

Physically, these acoustic oscillations correspond to the heating and ling of the fluid as it is compressed and rare-fied.

n order to determine $(\delta T/T)_k$ we need a theory of initial conditions. rom inflation we expect $(\delta T/T)_k(0)$ and $(\delta T/T)_k'(0)$ will be related the curvature perturbation R_k, which as we saw is linearly proportional Φ_k. In GR, the gravitational potential Φ is the time-time (00) uctuation of the metric, corresponding to a temporal shift $\frac{\delta t}{t} \sim \Phi$ nce the temperature $T \sim \frac{1}{a}$ and $a \sim t^{2/[3(1+w)]}$ the fractional hange in temperature is

$$\frac{\delta T}{T} = -\frac{\delta a}{a} = -\frac{2}{3}\frac{1}{1+w}\frac{\delta t}{t} = \begin{cases} -\frac{1}{2}\Phi & \text{RAD} \quad (w=1/3) \\ -\frac{2}{3}\Phi & \text{MAT} \quad (w=0) \end{cases}$$

nd through $\Phi_k = \frac{-3(1+w)}{5+3w} R_k$ gives the relationship for $\left(\frac{\delta T}{T}\right)_k$ nitial conditions to R_k. Now remember that for curvature dfiabatic) perturbations (as we assumed) R_k is conserved at large cale, therefore these initial conditions will correspond to selecting he cosine solution, i.e. $\left(\frac{\delta T}{T}\right)_k(0) \neq 0$ but $\left(\frac{\delta T}{T}\right)_k'(0) = 0$. Then we nave:

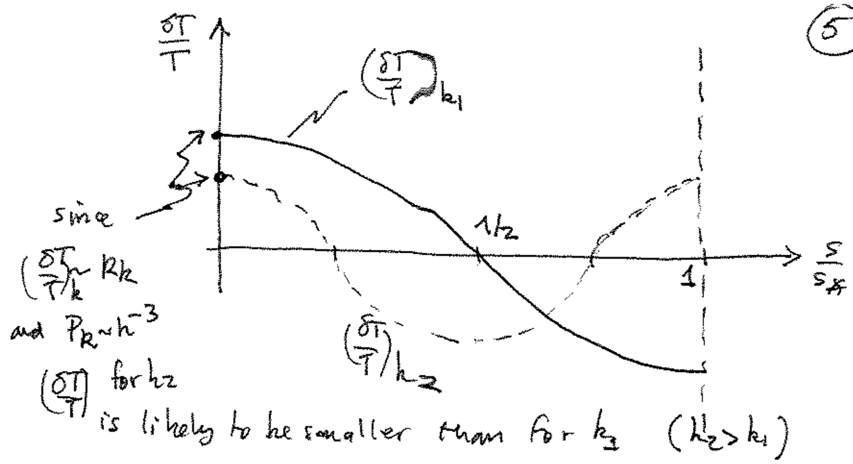
$$\left(\frac{\delta T}{T}\right)_k(\tau) = \left(\frac{\delta T}{T}\right)_k(0) \cos(ks) \qquad \text{with } \left(\frac{\delta T}{T}\right)_k(0) \propto R_k$$

hich at recombination $s \equiv s_*$ gives simply

$$\left(\frac{\delta T}{T}\right)_k^* = \left(\frac{\delta T}{T}\right)_k(0) \cos(ks_*)$$

Note that for $ks_* \ll 1$ $\cos(ks_*) \simeq 1$ and $\left(\frac{\delta T}{T}\right)_k^* \simeq \left(\frac{\delta T}{T}\right)_k(0)$ so at large scales we just see the primordial fluctuations, which from inflation should have approximate Harrison-Zel'dovich spectrum. Now, looking at this expression we see that at smaller scales the modes which are caught in the extrema of their oscillation will have enhanced fluctuations. Thus happens for :

$$k_n s_* = n\pi$$



he mode k_1 corresponds to a mode at completes half an oscillation , recombination, the second mode : corresponds to a full oscillation, etc. ince we are ~~...~~ interested in the ower spectrum, the main observable, we here

$$\left(\frac{\delta T}{T}\right)_k^2 \Bigg]_{@\,REC} = \left[\left(\frac{\delta T}{T}\right)_k (0)\right]^2 \cos^2(k s_*)$$

$\Rightarrow$ the spectrum will have peaks corresponding to multiples of $\frac{\pi}{s_*}$, a fundamental scale given by the sound horizon @ REC. These peaks will be harmonically related, in the sense that their separations will be in ratios 1:2:3....

In practice, as we'll discuss below, what we observe is not the 3D distribution $\left(\frac{\delta T}{T}\right)_k$ but rather its projection into the 2D sky as photons travel to us from the surface of last scattering (where they last scattered free electrons, right after REC (ignoring reionization much later)).
These peaks will appear at angular scales mapped from k_n by the angular diameter distance :

$$\begin{cases} \ell_1 \simeq k_1 d_A \\ \theta_1 \simeq \frac{2\pi}{\ell_1} \simeq \frac{2\pi}{k_1 d_A} \end{cases}$$

(ℓ: is Legendre multipole, since in 2D sky we decompose into spherical harmonics instead of Fourier modes, see below)

Where the angular diameter distance is (recall the FRW metric):

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 \underbrace{(d\theta^2 + \sin^2\theta\, d\phi^2)}_{\equiv d\alpha^2}\right] \equiv -dt^2 + a^2(t)\left[d\chi^2 + f^2(\chi)\, d^2\alpha\right]$$

comoving distance   comoving ang. angular diam.

where $\chi = \int \frac{dt}{a} = \tau$ for a radial geodesic light path

u comoving distance to REC is then : $\Delta\chi = \tau_0 - \tau_* \simeq \tau_0$

or the angular (comoving) distance we have $f(\chi) = \begin{cases} \sin\chi & k=1 \\ \chi & k=0 \\ \sinh\chi & k=-1 \end{cases}$

refore, for a flat universe $d_A = d_{comov} = \chi = \tau$, so

$$\boxed{\theta_1 \simeq \frac{2\pi}{\frac{\pi}{s_*} d_A} \underset{\uparrow}{\sim} \frac{2\pi}{\pi} \frac{\tau_*}{\sqrt{3}} \frac{1}{\tau_0} \sim \frac{\tau_*}{\tau_0} \underset{\substack{\uparrow \\ \text{MAT} \\ a \propto \tau^2}}{\sim} \sqrt{\frac{a_*}{a_0}} \underset{\substack{\uparrow \\ z_{REC} \sim 1000}}{\sim} \frac{1}{30} \rightarrow 2^\circ}$$

$s_* \sim \frac{\tau_*}{\sqrt{3}}$

or $\boxed{\ell_1 \sim \frac{2\pi}{\theta_1} \simeq 200}$ ← angular position of first "acoustic peak"

if the universe is not flat a given characteristic distance s_* corresponds to a different angular scale.

For an open universe the same s_* subtends a smaller angle :

⇒ first peak in power spectrum moves to higher ℓ's

open case ($k=-1$)

For a closed universe the angular scale corresponding to s_* is larger ⇒ first peak moves to lower ℓ's :

closed case ($k=+1$)

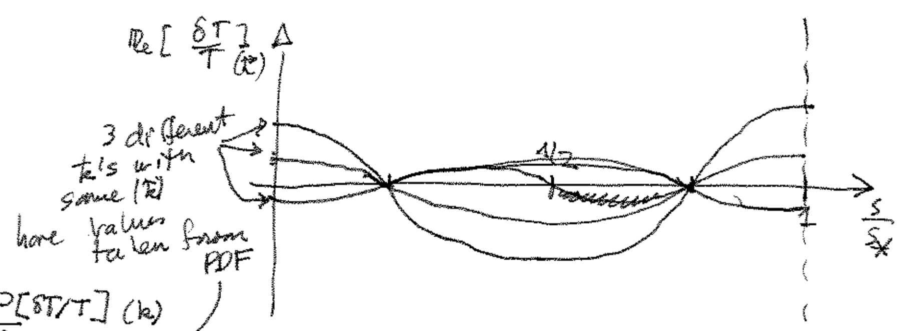
herefore, the sound horizon provides a fiducial scale (sometimes described as a "ruler" or "yardstick") that we can use to test the geometry of the universe! In practice this must be done more carefully than described here, e.g. s_* depends on baryon density and matter-radiation ratio so it is not exactly $\tau_*/\sqrt{3}$, and the angular diameter distance involves the Hubble constant which depends on dark energy density and its equation of state.

Another approximation done above to derive the angular position of acoustic peaks is to use that perturbations are purely in the adiabatic mode (cosine dependence is the only one excited). The opposite situation would be to excite the <u>isocurvature mode</u> for which $R_k = 0$ and $R_k' \neq 0$, corresponding to the sine solution. In this case acoustic oscillations lead to peaks at different positions corresponding to

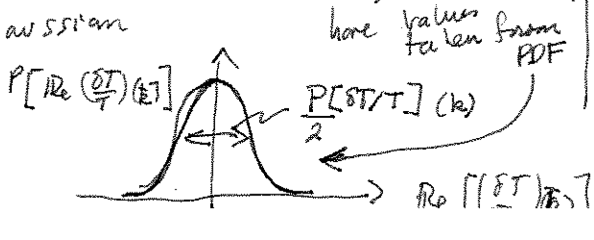
$$k_n s_* = (n \pm \tfrac{1}{2}) \pi \qquad \text{(isocurvature)}$$

Therefore, for given s_* these peaks will appear at different angular position if d_A is the same. Given the angular scale of peaks observed, which are consistent with pure adiabatic modes in flat universe, one would need to <u>drastically</u> change the angular diameter distance d_A by making the universe closed by a large Λ, so pure isocurvature modes are ruled out by CMB observations. <u>However</u>, if one considers an arbitrary admixture of adiabatic & isocurvature modes, as long as the dominant component is adiabatic things are observationally acceptable, and flatness is coupled to many parameters (cosmological) as well as initial conditions (e.g. % of isocurvature modes), so it is harder to say something conclusive.

Other models than inflation that generate fluctuations causally, such as phase transitions leading to <u>topological defects</u>, give quite different predictions. The reason is that in the case of inflationary curvature perturbations all k-modes have the same initial conditions, $(\frac{\delta T}{T})(\lambda) \neq 0$ and $(\delta T/T)'(\lambda) = 0$, independent of k. That means all the oscillators are in phase, e.g. for $\vec{k}$ modes with the same $|\vec{k}|$ we have the following behavior:



$(\frac{\delta T}{T})(\vec{k})$ can be written as a real & imaginary part, both of which are Gaussian fields so e.g.:



Although their Fourier coefficients are not the same (being taken out [8]
Gaussian random field with a Harrison-Zel'dovich power spectrum),
rent $\vec{k}$'s oscillate in phase so when $s = s_*$ @ REC all of them
have the same modulation of the initial fluctuations.

On the other hand in models where perturbations are generated
topological defects lead to phases that depend on k or even on $\vec{k}$ for
ne k, so one has $\cos[ks + \phi(\vec{k})]$ instead $\cos(ks)$. This
ds to temporal incoherence and the temperature fluctuations
$s = s_*$ are not simply related to the initial conditions, i.e.

$$\left|\left(\frac{\delta T}{T}\right)^*_k\right|^2 = \left|\sum_{\vec{k}} \left(\frac{\delta T}{T}\right)_{\vec{k}}(0) \cos\left[k s_* + \phi(\vec{k})\right]\right|^2$$

is not simply related to $\left|\sum_k \left(\frac{\delta T}{T}\right)_{\vec{k}}(0)\right|^2$

ike in the curvature fluctuations case:

$$\left|\left(\frac{\delta T}{T}\right)^*_k\right|^2 = \left|\sum_k \left(\frac{\delta T}{T}\right)_{\vec{k}}(0) \cos(k s_*)\right|^2 = \cos^2(k s_*) \left|\sum_k \left(\frac{\delta T}{T}\right)_{\vec{k}}(0)\right|^2$$

a result of this, in defect models, there are no acoustic peaks in the
mperature spectrum. The detection of acoustic peaks therefore rules out
viability of such models providing the dominant mechanism for
ctuation generation.

Now let's go back to the equation of motion for temperature fluctuations
d add the gravitational potential forcing (still keeping $R = 0$, i.e.
baryon inertia)

$$\left(\frac{\delta T}{T}\right)''_k + c_s^2 k^2 \left(\frac{\delta T}{T}\right)_k = -\frac{k^2}{3}\Phi_k$$

ecall that we are assuming MAT domination and Φ_k is indep of time in
is case. Since $c_s^2 = 1/3$ $(R=0)$ we can simply rewrite this equation
as:

$$\left(\frac{\delta T}{T} + \Phi\right)''_k + c_s^2 k^2 \left(\frac{\delta T}{T} + \Phi\right)_k = 0$$

Therefore, the solution is just an offset version of the previous
on with $\Phi_k = 0$:

$$\left(\frac{\delta T}{T} + \Phi\right)_k (\tau) = \left(\frac{\delta T}{T} + \Phi\right)_k (0) \quad \cos(ks)$$

that the effect of gravitational forcing is to change the zero-point
oscillation. As we discussed above $\left(\frac{\delta T}{T}\right)(0) = -\frac{2}{3}\Phi(0)$ during MAT
so we can rewrite this as:

$$\left(\frac{\delta T}{T}\right)^{eff}_k \equiv \left(\frac{\delta T}{T} + \Phi\right)_k (\tau) = \frac{1}{3} \quad \Phi_k(0) \quad \cos(ks)$$

(as usual, by $\tau = 0$ we mean when a given k mode crosses H^{-1} back, that's where we are imposing "initial conditions")

effective temperature thus oscillates about zero.
notice that this effective temperature is also the observed temperature fluctuation,
photons lose energy climbing out of gravitational potentials at REC.
fore, this is physically what we are after. Notice that in the
-scale limit $(ks \ll 1)$

$$\left(\frac{\delta T}{T}\right)^{eff}_k \simeq \frac{1}{3}\Phi_k(0)$$

is a famous result, the Sachs-Wolfe effect. It implies that overdense
ns (where $\Phi < 0$) correspond to cold spots in the temperature maps
photons climb out of the potential and get redshifted).

Now let's include the baryons. For simplicity we work in the adiabatic
roximation where $R'/R \ll k c_s$, so we can ignore the time dependence of the
on to photon momentum densities. We still work under the assumption of
tter domination where $\Phi' = 0$. Then we have

$$\left(\frac{\delta T}{T}\right)''_k + c_s^2 k^2 \left(\frac{\delta T}{T}\right)_k \simeq -\frac{k^2}{3}\Phi_k \qquad c_s^2 = \frac{1}{3}\frac{1}{1+R}$$

ch can be rewritten as

$$\left(\frac{\delta T}{T} + (1+R)\Phi\right)''_k + c_s^2 k^2 \left(\frac{\delta T}{T} + (1+R)\Phi\right)_k = 0$$

with solution

$$\left(\frac{\delta T}{T} + (1+R)\Phi\right)_k (\tau) = \left(\frac{\delta T}{T} + (1+R)\Phi\right)_k (0) \quad \cos(ks)$$

for the observed effective temperature:

$$\left(\frac{\delta T}{T}\right)_k^{\mathrm{eff}} (\tau) = \left(\frac{\delta T}{T} + \Phi\right)_k (\tau) = -R\,\Phi_{k(0)} + \left(\frac{\delta T}{T} + (1+R)\Phi_k\right)(0) \quad \cos(ks)$$

Φ indep of time

initial [conditions] $\frac{T}{T}(0) = -\frac{2}{3}\Phi$

$$\boxed{\left(\frac{\delta T}{T}\right)_k^{\mathrm{eff}} (\tau) = -R\,\Phi_{k(0)} + \left(R + \frac{1}{3}\right)\Phi_{k(0)} \quad \cos(ks)} \qquad s = \int_0^{\tau} \frac{d\tau}{\sqrt{3(1+R)}}$$

baryon-photon ratio R enters in three different ways

i) the soundspeed is decreased $\Rightarrow$ sound horizon @ rec s_* is smaller
$\Rightarrow$ peaks in l will move to the right $\quad l_1 \propto \sqrt{1+R}$

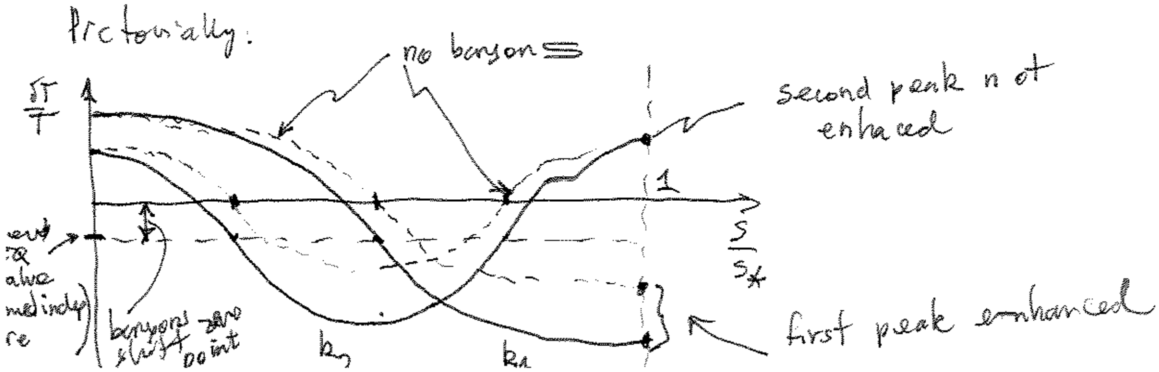
ii) the oscillation amplitude is enhanced by a factor $(1+3R)$

iii) the zero-point of oscillation is shifted, as expected since additional inertia of baryons means the equilibrium point of oscillator shifts in the gravitational field. This has important consequences, as the amplitude of the peaks is now modulated

$$\left| \left[\left(\frac{\delta T}{T}\right)_k^{\mathrm{eff}}\right]_{\text{odd peaks}} \right| = \left(2R + \frac{1}{3}\right)|\Phi_k| = (6R+1)\,\frac{1}{3}|\Phi_k|$$

$$\left| \left[\left(\frac{\delta T}{T}\right)_k^{\mathrm{eff}}\right]_{\text{even peaks}} \right| = \frac{1}{3}|\Phi_k|$$

Therefore baryons enhance every other peak, physically the extra gravity provided by the baryons enhance the compression into potential wells.
Pictorially:



We can write down the power spectrum of temperature fluctuations (11)
t recombination:

$$\left\langle \left| \left(\frac{\delta T}{T}\right)_k^{eff} \right|^2 \right\rangle_{REC} = |\Phi_k(0)|^2 \quad \left| \left(R+\frac{1}{3}\right) \cos(k s_*) - R \right|^2$$

is gives a basic picture of how acoustic oscillations will be reflected in e CMB spectrum. However, there are some important effects that we have eglected so far that give corrections to this:

1) Decays of potentials

As we mention at the beginning, the ratio of MAT to RAD is

$$\frac{\bar{\rho}_M}{\bar{\rho}_R} \simeq 3.6 \quad \frac{\Omega_m h^2}{0.15} \quad \frac{1000}{1+z}$$

so it is not completely negligible, in particular before REC. As we discussed when working out the transfer function, the potential Φ_k decays (when there is significant RAD) for scales less than H^{-1}. This has two effects: i) the time dependent decaying potential drives the oscillator since it is actually timed to leave the fluid maximally compressed with no gravitational potential to fight as it turns around, this enhances the peaks by about a factor of ~2 (in $\delta T/T$). This means that the alternating peak heights from the baryon inertia is not quite realized in practice. ii) because potentials are decaying there is an extra effect due to photons gaining energy as the drop into potential wells but not giving it back later when they climb out because potential decays. This is known as the Integrated Sachs-Wolfe effect (ISW). This "early" ISW amplifies the first peak, "early" because it is due to $\dot{\Phi}$ during RAD-MAT transition. There is also a "late" ISW effect due to dark energy, as photons travel to us. Again potentials decay due to acceleration of the universe, this enhances the low l part of the spectrum. before the first peak.

2) From k to l

We'll give expressions for this below, but since a given k-mode contributes to many l's there will be no zeros in the spectrum in l space, only some low values (troughs).

3) Doppler effect

So far we ignored that velocities of the baryon-photon fluid lead to Doppler shifts in $\delta T/T$ that we observe. This adds an "out of phase" component (since $v_\gamma \sim \sin(ks)$ for $\frac{\delta T}{T} \sim \cos(ks)$) and fills in the troughs and raises the overall fluctuation amplitude, decreasing the ratio of peak to trough amplitudes.

4) Damping

At small scales, where λ is comparable to photon mean free path, the fluid approximation breaks down. This leads to viscosity and heat conduction that damps acoustic oscillations, in practice this damps the 4th and higher peaks.

There are other interesting effects that we have no time to discuss, e.g. the impact of tensor modes, reionization, and polarization.

Now let's briefly discuss the mapping from k to l. We describe the temperature $T(\vec{x})$ by its Fourier transform:

$$T(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \, T(\vec{k}) \, e^{i\vec{k}\cdot\vec{x}}$$

with power spectrum

$$\langle T(\vec{k}) \, T(\vec{k}') \rangle = (2\pi)^3 \delta_D(\vec{k}+\vec{k}') \quad P_T(k)$$

(Note different Fourier convention here! to be consistent with CMB literature)

As usual the variance in real space is

$$\langle T^2(\vec{x}) \rangle = \int \frac{d^3k}{(2\pi)^3} P(k) = \int d\ln k \; \Delta_T(k)$$

where $\Delta_T(k) \equiv \dfrac{4\pi k^3 P_T(k)}{(2\pi)^3}$

The temperature field at $s = s_*$ that comes to us can be written as:

$$T(\hat{n}) = \int \frac{d^3k}{(2\pi)^3} \, T(\vec{k}) \, e^{i\vec{k}\cdot\hat{n}\, D_*}$$

where $D_* \simeq \tau - \tau_*$ is the comov. distance to the last-scattering surface (assume flat univ.)

Expanding the plane wave in spherical coordinates (Rayleigh expansion):

$$e^{i\vec{k}\cdot\hat{n}\, D_*} = 4\pi \sum_{l,m} i^l \, j_l(k D_*) \, Y^*_{lm}(\hat{k}) \, Y_{lm}(\hat{n})$$

we have the multipole moments T_{lm}

$$T_{lm} = \int \frac{d^3k}{(2\pi)^3} \, T(\vec{k}) \, 4\pi \, i^l \, j_l(kD_*) \, Y_{lm}(\hat{k})$$

with power spectra:

$$\langle T^*_{lm} T_{l'm'} \rangle = \int \frac{d^3k}{(2\pi)^3} (4\pi)^2 \, i^{l-l'} \, j_l(kD_*) \, j_{l'}(kD_*) \, Y^*_{lm}(\hat{k}) \, Y_{l'm'}(\hat{k}) \, P_T(k)$$

$$= \delta_{ll'} \, \delta_{mm'} \, 4\pi \int d\ln k \; j_l^2(kD_*) \, \Delta_T(k)$$

For a slowly varying $\Delta_T(k)$ (remember in HZ case $\Delta_T(k) = \text{const.}$), use

$$\int_0^\infty j_l^2(x) \, d\ln x = \frac{1}{2l(l+1)}$$, we have:

$$C_l \simeq \frac{4\pi \, \Delta_T(l/D_*)}{2l\,(l+1)} = \frac{2\pi}{l(l+1)} \, \Delta_T(l/D_*)$$

That's why plots usually show $\frac{l(l+1)}{2\pi} C_l = \Delta_T(l/D_*)$ is the dimensionless power spectrum.

[Show results]