Distribution Functions

We will now start discussing the thermal history of the universe. Before we do so, let's review some basics of units that we are going to use from now on:

In CGS, $\hbar = 1.05 \times 10^{-27}$, $c = 2.99 \times 10^{10}$ and $k_B = 1.38 \times 10^{-16}$

It is customary in cosmology, when studying the early universe, to work with units such that

$$\hbar = c = k_B = 1 \qquad \text{("natural units")}$$

Therefore, energy becomes the only dimensional quantity, e.g.

$$[\text{Energy}] = [\text{Mass}] = [\text{Temperature}] = [\text{Length}]^{-1} = [\text{Time}]^{-1}$$

For example:

$$1\,\text{GeV} = 10^9\,\text{eV} = 1.6 \times 10^{-3}\,\text{erg}$$
$$= 1.16 \times 10^{13}\,{}^\circ\text{K}$$
$$= 1.78 \times 10^{-24}\,\text{g}$$

and

$$1\,\text{GeV}^{-1} = 1.97 \times 10^{-14}\,\text{cm}$$
$$= 6.58 \times 10^{-25}\,\text{sec}$$

Using this it is easy to see that

$$\rho_{crit} = 1.879\, h^2 \times 10^{-29} \frac{\text{g}}{\text{cm}^3} = 8 h^2 \times 10^{-47}\,\text{GeV}^4 = h^2\, 10^4 \frac{\text{eV}}{\text{cm}^3}$$

For Newton's gravitational constant we have

$$G = 6.67 \times 10^{-8} \quad \text{in CGS}$$

From the fundamental constants we can define the characteristic energy or mass, known as Planck mass m_{Pl}

$$m_{Pl} = \sqrt{\frac{\hbar c^5}{G}} \underset{\hbar = c = 1}{\equiv} G^{-1/2} = 1.22 \times 10^{19}\,\text{GeV}$$

Then, the Friedmann equation at early times can be written as

$$H^2 = \frac{8\pi G}{3}\rho \sim \frac{\rho}{m_{Pl}^2} \qquad \text{in natural units}$$

We will also deal with the weak interaction, characterized by Fermi's constant G_F:

$$G_F = 1.16 \times 10^{-5}\ \text{GeV}^{-2} = 1.02 \times 10^{-5}\ m_p^{-2}$$

proton mass

Similarly, electromagnetic interactions are characterized by the fine-structure constant α

$$\alpha = \frac{e^2}{\hbar c\, 4\pi\epsilon_0} = \frac{e^2}{4\pi} \sim \frac{1}{137} = 7.3 \times 10^{-3}$$

You can check appendix A from Kolb & Turner for many of these relations.

Distribution functions.

The early universe was to a very good approximation, as we shall see, in thermal equilibrium. Depending on the temperature T, the thermal bath will be populated by different kinds of elementary particles at different times.

For example, when $T \gg m_e$, lots of electrons and positrons must be around due to $\gamma + \gamma \to e^+ + e^-$, these will be ultra relativistic since $T \gg m$.

To describe the different components of the thermal bath we use phase-space distribution function $f(\vec{p}, t)$ which describes for each species how many particles per unit momentum and volume are there at each time. [Note that f does not depend on $\vec{x}$ due to homogeneity, and by isotropy will depend only on $|\vec{p}|$]

For a weakly interacting gas of particles with g internal degrees of freedom we have:

number density : $n = \frac{g}{(2\pi)^3} \int f(\vec{p})\, d^3p$

energy density : $\rho = \frac{g}{(2\pi)^3} \int E(\vec{p})\, f(\vec{p})\, d^3p$

pressure : $p = \frac{g}{(2\pi)^3} \int \frac{p^2}{3E(\vec{p})}\, f(\vec{p})\, d^3p$

$\gamma = (1-v^2)^{-1/2}$

$E = m\gamma$

$p = mv\gamma$

where $E^2(\vec{p}) = p^2c^2 + m^2c^4 \to p^2 + m^2$ in natural units.

for the pressure, note that for a relativistic fluid $p = \frac{1}{3} m v^2 \gamma \overset{\downarrow}{=} \frac{p^2}{3E}$

We will use p for pressure, $\mathbf{p}$ for momentum (should be distinguishable!)

For elementary particles g is the spin degeneracy, i.e. $g = 2J+1$ for spin J, but keep in mind that

i) for photons (γ's), $J=1$ but $g=2$ (because photons are transverse)

ii) neutrinos (ν's), $J = 1/2$ since there is only one helicity (in STD model), $g = 1$

iii) electrons, positrons, $J = 1/2$ $g = 2J+1 = 2$ ✓

If the interactions are "fast enough" (we'll define this shortly), they can maintain thermal equilibrium, then we have

$$f_i(\vec{p},t) \simeq \cancel{} \left[\exp\left[\frac{E_i - \mu_i}{T(t)}\right] \pm 1 \right]^{-1} \qquad \begin{array}{l} + : \text{FD (fermions)} \\ - : \text{BE (bosons)} \end{array}$$

"species i"

where $T(t)$ is the instantaneous temperature, $E_i^2 = p_i^2 + m_i^2$, and μ_i is the chemical potential for species i.

In chemical equilibrium when species i,j interact to give k,l the individual number of particles in each species is not conserved but the total is, then

$$i + j \leftrightarrow k + l \quad \Rightarrow \quad \mu_i + \mu_j = \mu_k + \mu_l$$

The similarity with heat and temperature may be useful here. If we have two systems with T_1 and T_2, and if $T_2 > T_1$ then we know heat will flow from 2 to 1 when they are brought in contact. Similarly when 2 species with different μ's are interacting the one with higher μ will give up particles to the other.

Note that

i) Photons can be absorbed and emitted in arbitrary numbers, so $\mu_\gamma = 0$ (Planck spectrum)

ii) particle-antiparticle can annihilate into γ's

$\Rightarrow \mu_{e^+} = -\mu_{e^-}$, etc

In general we will work with $\mu_i = 0$ (good enough @ high T, but we will see later $\mu_i \neq 0$ in e.g. BBN).

Note that as $T \to 0$ ($\mu/T \to \infty$) for fermions we recover a very special distribution function

$$f(\vec{p}) = \begin{cases} 1 & \text{if } E \leq \mu \equiv E_F \\ 0 & \text{if } E > \mu \equiv E_F \end{cases}$$

where the <u>Fermi Energy</u> is the chemical potential μ @ $T=0$. In such cases, when Pauli's principle becomes important, it is said that ~~the~~ the system is <u>degenerate</u>.

<u>Thermal Equilibrium</u>

We mentioned that thermal equilibrium is achieved when interactions are "fast enough". Fast enough compared to what? ~~The~~ characteristic time scale is H^{-1} given by the expansion of the universe.

A simple criterion that determines whether equilibrium holds is to require that the interaction rate Γ (i.e. collisions per unit time) be large ~~[illegible]~~ compared to the expansion rate H:

$$\text{if} \quad \Gamma \gtrsim H \quad \Rightarrow \quad \text{EQUILIBRIUM}$$

where $\Gamma \equiv \underset{\substack{\uparrow \\ \text{number} \\ \text{density}}}{n} \underbrace{\langle \sigma \, v \rangle}_{\substack{\uparrow \\ \text{interaction cross section} \\ \text{averaged over energy}}}$

relative velocity ~ typical v's

All species that satisfy $\Gamma_i \gtrsim H$ will be in equilibrium with $f_i(\bar{p}, t)$ evolving adiabatically with $T(t)$, and they will all share the same temperature. When $\Gamma_i < H$ we say that species i has <u>decoupled</u>.

In order to see how the equilibrium conditions depend on temperature, let's study how H and Γ depend on temperature T.

First, let's do a simplified analysis to find $H(T)$. If we take the universe as RAD dominated early on we see that

$$H^2 = \frac{8\pi G}{3} \rho_R \approx \frac{\rho_R}{M_{pl}^2} \underset{\substack{\uparrow \\ \text{for a black body} \\ \rho_R \sim \sigma T^4}}{\sim} \frac{T^4}{m_{pl}^2} \qquad \text{but} \quad \rho_R \sim \frac{1}{a^4}$$

$$\Rightarrow T \sim \frac{1}{a}$$

Although this is approximate (because the number of relativistic species does change as the universe expands) it is a good first approximation. The point is what we mean by "radiation". In fact what we mean is all the species that when temperature is T they have masses $m_i \ll T$ (so they are effectively massless) and they have interactions fast enough so they are in equilibrium. We'll do a better job shortly. For now let's take $T \sim \frac{1}{a}$.

$$\Rightarrow \quad H(T) \simeq \frac{T^2}{m_{pl}}$$

To find $\Gamma(T)$ we need to know:

$$n \sim \frac{1}{a^3} \sim T^3$$

How about $\sigma(T)$? This of course will depend on the type of interaction.

A little bit of field theory is needed to explain this fact. If you want, you can skip what follows and go straight to i) and ii) below and accept that's the way it is.

In field theory the way one think of forces is by exchange of some particle (boson), with mass m_X, and cross section is given by (as a function of energy)

$$\sigma_{(E)} \sim E^2 |A|^2$$

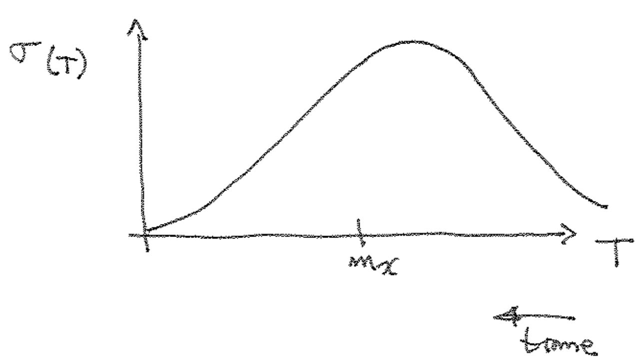
where A is the amplitude for the exchange, in terms of Feynman diagrams

$A_{(p)} \approx$  $\sim \dfrac{\lambda^2}{p^2+m_X^2}$

where λ is the vertex, λ^2 is basically the coupling constant of the interaction.

$$\Rightarrow \quad \sigma \sim \frac{E^2 \lambda^4}{(p^2+m_X^2)^2} \sim \begin{cases} \dfrac{\lambda^4 E^2}{m_X^2} \sim T^2 & \text{if } m_X \gtrsim T \\[2ex] \dfrac{\lambda^4}{E^2} \sim T^{-2} & \text{if } m_X \ll T \end{cases}$$

typical behavior $\Rightarrow$



As a result of this we have:

i) Mediation by a massless boson (e.g. the photon)

$$\sigma \sim \frac{\alpha^2}{T^2} \quad \Rightarrow \quad n\langle\sigma v\rangle \underset{v\sim c=1}{\sim} \alpha^2 T$$

(here in this case) $\alpha \sim \lambda^2$

Then the condition for equilibrium is

$$\Gamma \sim \alpha^2 T \quad \gtrsim \quad H \sim \frac{T^2}{m_{pl}}$$

$\Rightarrow$ EQ holds for $\boxed{T \lesssim \alpha^2 m_{pl} \sim 10^{16}\ \mathrm{GeV}}$

ii) mediation by a massive boson with $m_X \gtrsim T$ (otherwise it would be effectively massless and go back to i))

$$\Gamma \sim G_X^2\, T^2 \qquad \text{with } G_X \sim \frac{\alpha}{m_X^2} \sim \frac{\lambda^2}{m_X^2}$$

$$\Rightarrow \quad \Gamma \sim G_X^2\, T^5 \quad \gtrsim \quad H \sim \frac{T^2}{m_{pl}}$$

$\Rightarrow$ EQ holds for $\boxed{m_X \gtrsim T \gtrsim G_X^{-2/3}\, m_{pl}^{-1/3} \sim \left(\frac{m_X}{100\ \mathrm{GeV}}\right)^{4/3}\ \mathrm{MeV}}$

Those are the conditions for thermal equilibrium.

Now let's consider limiting behavior of n, ρ, p from the general case given before:

a) Relativistic limit, $T \gg m$ (with $\mu \ll T$)

$$n \simeq \begin{Bmatrix} 1 \\ 3/4 \end{Bmatrix} \times \frac{\zeta(3)}{\pi^2}\, g\, T^3 \qquad \begin{Bmatrix} BE \\ FD \end{Bmatrix} \qquad \zeta(3) \sim 1.2$$

$$\rho \simeq \begin{Bmatrix} 1 \\ 7/8 \end{Bmatrix} \times \frac{\pi^2}{30}\, g\, T^4 \qquad \begin{Bmatrix} BE \\ FD \end{Bmatrix}$$

$$p \simeq \frac{1}{3}\rho \qquad \text{(as it should be)}$$

(Note that $\rho_\gamma = \sigma T^4$ for photons is recovered)

Stephan-Boltzmann constant

b) Non-Relativistic limit, $T \ll m$

$$n \simeq g \left(\frac{mT}{2\pi}\right)^{3/2} e^{-\frac{(m-\mu)}{T}}$$

$$\rho \simeq m\, n$$

$$p = nT \ll \rho$$

(then $p \simeq 0$) in relativistic terms

this is the usual Boltzmann distribution. You see that the bath with energy $E \sim k_B T$ cannot create m from plasma, and thus abundance is Boltzmann suppressed (note this is in EQ)

Now, at a given temperature (large enough) the energy density will be dominated by the "radiation", meaning those "degrees of freedom" (particles) that are massless, (relativistic) then effectively radiation energy is given by

$$\rho_R \equiv \frac{\pi^2}{30} g_* T^4 \qquad \left(p_R = \frac{1}{3}\rho_R\right)$$

where g_* is the effective number of massless degrees of freedom,

$$g_* = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_{\text{fermions}} g_i \left(\frac{T_i}{T}\right)^4 \underset{\text{if } T_i = T}{=} \sum_b g_i + \sum_f \frac{7}{8} g_i$$

Where we allowed for each species i to have in pple a different temperature T_i. The reason is that ~~they~~ some of them may have decoupled already, but they maintain a thermal distribution with a different $T_i \neq T$ from the thermal bath. We shall explain this soon.

Note: g_* is a function of T, since we sum over relativistic degrees of freedom, where $m_i \ll T$. As T is decreased, less particles can be considered massless and g_* decreases (at high enough T all particles are effectively massless and g_* is maximum).

For example, @ $T \ll 1$ MeV only photons and the 3 generations of neutrinos and antineutrinos are relativistic, then:

$$g_* = \underbrace{2}_{g \text{ for } \gamma\text{'s}} + \underbrace{3}_{\#\text{ of families}} \times \underset{\substack{\uparrow \\ \nu\text{'s are fermions}}}{\frac{7}{8}} \times \underset{\substack{\uparrow \\ g=1}}{1} \times \underset{\substack{\uparrow \\ \nu,\bar{\nu}}}{2} \times \underbrace{\left(\frac{4}{11}\right)^{4/3}}_{\downarrow \; \left(\frac{T_\nu}{T}\right)^4} \simeq 3.36$$

(we'll derive this later by entropy conservation)

ν's are decoupled already for $T \ll 1$ MeV

When $T \gtrsim 1$ MeV, neutrinos are in equilibrium, $T_\nu = T$ and also e^+, e^- are relativistic so there is an increase in g_* due to both effects

$$\Delta g_*^{e^+e^-} = \frac{7}{8} \times \underset{\substack{\uparrow \\ g=2}}{2} \times \underset{\substack{\uparrow \\ e^+, e^-}}{2} = 3.5$$

$$\Delta g_*^{\nu} = 3 \times \frac{7}{8} \times 2 \simeq 5.25$$

instead of what is written above

$$\Rightarrow g_* \approx 2 + 3.5 + 5.25 = 10.75$$

Brief description of Standard Model particles (see App. B in K&T for more)

In order to have a rough idea of the relativistic degrees of freedom as a function of temperature, we need masses and spins for particle content in the standard model of particle physics. Here is a very brief summary:

LEPTONS
$$\begin{cases} m_e \simeq 0.5 \text{ MeV} \\ m_\mu \simeq 100 \text{ MeV} \\ m_\tau \simeq 1.8 \text{ GeV} \end{cases}$$

$m_{\nu_e}, m_{\nu_\mu}, m_{\nu_\tau} = 0$ in std. model though we know from experiments e.g. ν oscillations from sun, atmospheric experiments, that they should have (small) masses

QUARKS
$$\begin{cases} m_u = 6 \text{ MeV} \\ m_d = 10 \text{ MeV} \\ m_s = 0.25 \text{ GeV} \\ m_c = 1.2 \text{ GeV} \\ m_b = 4.3 \text{ GeV} \\ m_t = 180 \text{ GeV} \end{cases}$$

$m_p = 938.272$ MeV

$m_n = m_p + 1.293$ MeV

Leptons and Quarks are arranged into 3 families:

g   Leptons (fermions, spin 1/2)   interact electroweak (electromagnetism, + weak force)

$$\begin{array}{cccc} 2 & \begin{pmatrix} e \\ \nu_e \end{pmatrix} & \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix} & \begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix} \\ 1 & & & \end{array}$$

Quarks (charge +2/3 and −1/3), spin 1/2

$$2 \quad \begin{pmatrix} u \\ d \end{pmatrix} \quad \begin{pmatrix} c \\ s \end{pmatrix} \quad \begin{pmatrix} t \\ b \end{pmatrix} \quad \begin{matrix} +2/3 \\ -1/3 \end{matrix}$$

interact via strong & electroweak

each quark comes in 3 colors (a kind of "charge" in QCD)

2   γ $m_\gamma = 0$ boson, spin 1   mediate electromagnetism

3    $W^\pm$ $m_W \simeq 80\,\mathrm{GeV}$ "   "   "   mediate weak interactions

3   Z^0 $m_Z \simeq 90\,\mathrm{GeV}$ "   "   "   "   "   "

2   8 Gluons $m = 0$ "   "   "   mediate color interaction

1   Higgs $115\,\mathrm{GeV} \lesssim m_H \lesssim 200\,\mathrm{GeV}$ "   "   0   (mechanism by which all particles above acquire their mass)

Note that for $T \gg 200\,\mathrm{GeV}$ these particles are all relativistic, then

$$g_*\Big]_{T\gg 200\,\mathrm{GeV}} = \overbrace{\underbrace{1}_{\text{Higgs}} + \underbrace{8\times 2}_{\text{Gluons}} + \underbrace{3\times 3}_{Z,W^\pm} + \underbrace{2}_{\gamma}}^{\text{bosons}} + \overbrace{\underbrace{2}_{q,\bar q} \times \underbrace{6}_{\#\text{ of quarks}} \times \underbrace{2}_{g} \times \underbrace{3}_{\#\text{ of colors}} \times \frac{7}{8}}^{\text{Quarks}} \;\leftarrow \text{fermions} \; +$$

$$+ \overbrace{\underbrace{2}_{l,\bar l} \times \left(3 \times 1 + 3 \times 2\right) \times \frac{7}{8}}^{\text{Leptons}} \;\leftarrow \text{fermions}$$

\# of families g_ν's $g_{e,\mu,\tau}$

$$\simeq 106.75$$