Conservation of entropy provides a powerful way of constraining the evolution of the temperature of particles in equilibrium.

From the second law applied to a physical volume $V = a^3$ we have

$$T\,dS = d(\rho_{eq} V) + p_{eq}\, dV$$

where "eq" here is to remind you (that is for equilibrium particles at temperature T, will skip ~~...~~ subscripts below)

$$\Rightarrow \begin{cases} \left.\dfrac{\partial S}{\partial V}\right]_{V,T} = \dfrac{\rho + p}{T} \\[2ex] \left.\dfrac{\partial S}{\partial T}\right]_{V,T} = \dfrac{V}{T}\dfrac{d\rho}{dT} \end{cases}$$

Now, the integrability condition that makes S an exact differential is

$$\frac{\partial^2 S}{\partial V \partial T} = \frac{\partial^2 S}{\partial T \partial V}, \quad \text{which implies}$$

$$\frac{\partial}{\partial T}\left[\frac{\rho+p}{T}\right] = \frac{\partial}{\partial V}\left[\frac{V}{T}\frac{d\rho}{dT}\right] \Rightarrow \frac{1}{T}\left(\cancel{\frac{d\rho}{dT}} + \frac{dp}{dT}\right) = \frac{\rho+p}{T^2} + \cancel{\frac{1}{T}\frac{d\rho}{dT}}$$

$$\Rightarrow \quad \frac{dp}{dT} = \frac{\rho+p}{T}$$

Now $\quad T\,dS = d\left[(\rho+p)V\right] - V\,dp \quad \Rightarrow \quad dS = \dfrac{d\left[(\rho+p)V\right]}{T} - \dfrac{V\,dp}{T}$

$$= \frac{d\left[(\rho+p)V\right]}{T} - V\frac{(\rho+p)}{T^2}\,dT = d\left[\frac{(\rho+p)V}{T}\right]$$

$$\Rightarrow \quad dS = d\left[\frac{(\rho+p)V}{T}\right]$$

Now, recall that from stress-energy conservation $\nabla_\nu(T^{\mu\nu}) = 0 \Rightarrow d(\rho V) = -p\,dV$

$\Rightarrow dS = 0$, then entropy is conserved, therefore up to an arbitrary additive constant

$$S = \frac{(\rho+p)V}{T} \quad \text{is conserved.}$$

– Note that in practice there are some small departures from entropy conservation, i.e. $\nabla_\nu(T^{\mu\nu})$ is not zero in short periods, but the ΔS is very small compared to $S_{photons}$, so negligible the entropy density is given by

$$s = \frac{S}{V} = \frac{\rho + p}{T}$$

From this we see that the entropy is dominated by the relativistic particles, then to a very good approximation

$$\boxed{s = \frac{\pi^2}{30} \frac{T^{\cancel{4}\,3}}{\cancel{T}} \left(1+\frac{1}{3}\right) g_*^s = \frac{2\pi^2}{45} g_*^s T^3}$$

where we defined $\quad g_*^s \equiv \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \sum_{\text{fermions}} \frac{7}{8} g_i \left(\frac{T_i}{T}\right)^3$

again, keeping in mind that not all relativistic particles may share the same T (some of them may have decoupled already, as we shall see later) - If $T_i = T$ then $g_*^s = g_*$ - Today $g_*^s \neq g_*$; for $t < 1$ sec $g_*^s \simeq g_*$ -

From this we derive the important consequence,

S is conserved $\Rightarrow$ $\boxed{S \propto g_*^s T^3 a^3 = \text{const.}}$ as the universe expands

$$\Rightarrow \quad T \propto (g_*^s)^{-1/3} \frac{1}{a}$$

During periods where $g_*^s = $ const. , this recovers the approximate $T \propto 1/a$ that we used before. Note that when a particle becomes NR (as its mass becomes comparable to T) , g_*^s drops , and therefore T drops <u>slower</u> than $1/a$ - The reason is that the entropy of these particles gets picked up by the other particles in the bath (the same S is shared among smaller g_*^s) , so effectively they don't cool as fast as $1/a$ -

Entropy conservation tells us how particles that belong to the thermal bath cool off as the universe expands - However, how do the distribution of decoupled particles evolve once they decoupled and thus fall out of thermal equilibrium?

Their evolution can be obtained from the fact that after decoupling a species evolves without interacting and therefore its distribution function $f(\vec{p},t) \equiv \frac{dn}{d^3p}$ is conserved. Then we know that $n \propto a^{-3}$ and therefore $p \sim \frac{1}{a}$ (this is completely analogous to the result for photons, where $E_\gamma \sim \frac{1}{a}$, and could be derived from the geodesic equation). Therefore, after decoupling we have

$$f(\vec{p},t) = f_{eq}\left(\vec{p}(t)\frac{a(t)}{a_{dec}}, t_{dec}\right) \qquad t > t_{dec}$$

from here we can derive the evolution of temperature for decoupled species. We have two cases, as usual

i) Relativistic while it decouples $\mu, m \ll T_{dec}$

$$\Rightarrow f(p,t) = f_{eq}\left(p(t)\frac{a}{a_{dec}}, t_{dec}\right) \sim \left\{\exp\left[\frac{p\,a/a_{dec}}{T_{dec}}\right] \pm 1\right\}^{-1}$$

Notably, this is an equilibrium distribution, with an effective temperature

$$\boxed{T \equiv T_{dec}\frac{a_{dec}}{a} \sim \frac{1}{a}}$$

Therefore, a species that decouples while relativistic evolves afterwards with a temperature $T \sim 1/a$. From this we see that this species separately conserves its entropy, since $s \sim T^3 \sim \frac{1}{a^3} \Rightarrow S = s a^3 = \text{const.}$

From this distribution function we readily obtain,

$$n = \left\{\begin{matrix} 1 \\ 3/4 \end{matrix}\right\} \frac{\zeta(3)}{\pi^2}\, g\, T_{dec}^3 \left(\frac{a_{dec}}{a}\right)^3$$

going as $1/a^3$. This gives the number density after decoupling. Note that such species has a very similar number density to those in photons, so it will be around today (as e.g. the neutrino background).

As the universe expands, if m is not exactly zero, it will eventually reach a point when $m \sim T$, so the decoupled particles become non relativistic. However, the distribution function is still given by that of the relativistic case, as it must be conserved. The only difference is that now the energy $E \sim m$ and therefore the number density and energy density are related by $\rho = m\, n$.

ii) Non-relativistic @ decoupling, $m \gg T_{dec}$ $(\mu \ll m)$

Then we have:

$$f(p,t) = f_{eq}\left(p(t)\frac{a}{a_{dec}}, t_{dec}\right) \underset{\substack{\uparrow \\ E \sim p^2/2m + m \\ \mu \approx 0}}{\simeq} \exp\left[-\frac{m + \frac{p^2}{2m}a^2/a_{dec}^2}{T_{dec}}\right]$$

We can reinterpret this ~~~~ ~~~~ from an equilibrium perspective

$$\frac{m + \frac{p^2}{2m}a^2/a_{dec}^2}{T_{dec}} \equiv \frac{m + p^2/2m - \mu}{T}$$

from $$\frac{\frac{p^2}{2m}\frac{a^2}{a_{dec}^2}}{T_{dec}} = \frac{p^2}{2m}\frac{1}{T} \Rightarrow \boxed{T = T_{dec}\left(\frac{a_{dec}}{a}\right)^2 \sim \frac{1}{a^2}}$$

$$\Rightarrow \frac{m}{T_{dec}} = \frac{m-\mu}{T} = \frac{m-\mu}{T_{dec}}\left(\frac{a}{a_{dec}}\right)^2 \Rightarrow \mu = m\left[1 - \left(\frac{a_{dec}}{a}\right)^2\right]$$

Therefore, after decoupling a NR species can be thought as having an equilibrium distribution with $T \sim 1/a^2$ and the above chemical potential. Note that the number density that follows from this is

$$n \sim g\left(\frac{mT}{2\pi}\right)^{3/2} \underbrace{\exp\left[-\frac{(m-\mu)}{T}\right]}_{\substack{\text{constant from} \\ \text{expression above} \\ \text{for } \mu, T}} \sim \frac{1}{a^3} \quad \checkmark$$

as it must be.

Relic background of relativistic particles : ν decoupling , MAT-RAD EQ, e^+e^- ANNH

The arguments above tell us generically how species evolved after they decouple (in the limits $m \ll T_{dec}$ and $m \gg T_{dec}$, when $m \sim T_{dec}$ things are more complicated and cannot be mapped to equilibrium distributions). In order to know which case corresponds to a particular species we have to consider the mass spectrum and interactions in the standard model of elementary particles.

To describe what's going on at temperature T we need to understand physics at energies $E \sim T$. For $T \lesssim 0.1$ TeV (~ 1 TeV soon) we have accelerators and understand the physics quite well. For $T \gtrsim 1$ TeV our predictions are based on extrapolations, particularly for $T \gtrsim 10^{16}$ GeV things become very uncertain, e.g naively @ 10^{19} GeV (Planck scale) classical general relativity is not appropriate anymore and one must restort to quantum gravity (this may actually happen at much lower energies according to some modern ideas).

But for $T \lesssim 1$ GeV $\sim 10^{13}$ K things are very well understood, and this is where the standard hot big bang model operates.

Before we get into specific interactions, let us write the expansion rate in a useful way. For RAD era we have

$$\left(\frac{\dot a}{a}\right)^2 = H^2 = \frac{1}{4t^2} = \frac{8\pi G}{3}\rho_R = \frac{8\pi G}{3} g_* \frac{\pi^2}{30} T^4$$

$$\Rightarrow \boxed{H = g_*^{1/2} T^2 \sqrt{\frac{8\pi^3 G}{90}} = 1.66\sqrt{G}\, g_*^{1/2} T^2 = 1.66\, g_*^{1/2}\left(\frac{T^2}{m_{pl}}\right)}$$

Also : $$t \simeq 0.3\, g_*^{-1/2}\left(\frac{m_{pl}}{T^2}\right) = 0.3\, g_*^{-1/2}\left(\frac{\text{MeV}}{T}\right)^2 \underbrace{\frac{1.22\times10^{22}\ \text{MeV}}{\text{MeV}^2}\ \frac{6.58\times10^{-25}\,\text{s}}{10^{-3}\ \text{MeV}^{-1}}}_{\simeq 8\ \text{sec.}}$$

$$\Rightarrow \boxed{t \simeq 2.4\, g_*^{-1/2}\left(\frac{\text{MeV}}{T}\right)^2 \text{sec}}$$

Remember $g_* \simeq 10.75$ for $T \gtrsim 1$ MeV
$\Rightarrow$ $t \sim 1$ sec @ $T = 1$ MeV

Let's begin the discussion of the thermal history of the universe at $T \sim 10^{12}\,K \sim 85$ MeV. At this temperature the only relativistic degrees of freedom are: photons, neutrinos and electrons and positrons (recall $m_e \sim 0.5$ MeV). We are at or below the $W^{\pm}, Z^0$ masses, and all quarks are already inside neutrons and protons (the quark-hadron phase transition happens at about 0.2 GeV). We will assume all chemical potentials are zero. Then we have:

$e, \bar{e}, \nu, \bar{\nu}, \gamma$'s in EQ all at same T, then

$$\boxed{g_* = g_*^S = g_{bosons} + \frac{7}{8} g_{fermions} = 2 + \frac{7}{8}\left[2\times 2 + 3\times 2\right] = \frac{43}{4} = 10.75}$$

$$\Rightarrow \boxed{H(T) = 1.66\, g_*^{-1/2} \frac{T^2}{m_{pe}} = 5.4 \frac{T^2}{m_{pe}}}$$

Neutrinos have no electric charge and they interact by weak interactions with electrons and positrons, and equilibrium is maintained by

$$\nu\bar{\nu} \underset{Z^0}{\longleftrightarrow} e\bar{e}, \qquad \nu e \underset{Z^0}{\longleftrightarrow} \nu e$$

the cross section is, as we discussed, roughly

$$\sigma(E) \sim E^2 |A|^2 \sim E^2 \left(\frac{\lambda^2}{p^2 + m_Z^2}\right)^2 \underset{p \sim E \lesssim m_Z \sim 90\,\text{GeV}}{\sim} E^2 \frac{\lambda^4}{m_Z^4} \equiv E^2 G_F^2 = G_F^2 T^2$$

then, the interaction rate is

$$\Gamma = n\langle \sigma v \rangle \underset{v \sim c = 1}{\sim} \frac{\zeta(3)\, g_*}{\pi^2} T^3 \; G_F^2 T^2 \sim 1.3\, G_F^2 T^5$$

$$\Rightarrow \boxed{\frac{\Gamma}{H} \simeq \frac{1}{4} T^3 \frac{m_{pe}}{G_F^{-2}} \sim \left(\frac{T}{1.4\,\text{MeV}}\right)^3 = \left(\frac{T}{1.6\times 10^{10}\,K}\right)^3}$$

Therefore, ν's can be in EQ until $T \sim 1.4$ MeV $\sim 1.6\times 10^{10}$ K. At this time $T_\nu = T_\gamma$, afterwards $T_\nu \sim 1/a$, whereas for photons and electrons (while relativistic) also $T \sim 1/a$. However, eventually when

T reaches 0.5 MeV, the e^+, e^- become NR and transfer their entropy to the photons, which makes them cool slightly slower. The neutrinos do not care since they've decoupled already and thus their entropy is separately conserved. Let's estimate what's going here. The idea is that as T approaches 0.5 MeV we have

$$e^+e^- \longrightarrow \gamma\gamma$$ [with γ's getting thermalized by Compton scattering]

but the inverse process

$$\gamma\gamma \longrightarrow e^+e^-$$ becomes suppressed

Therefore the number density $n_{e^+} n_{e^-}$ starts to drop exponentially (which is what means to be in EQ when a species becomes NR).

As the e^+e^- become NR they cease to contribute as Relativistic degrees of freedom, this leads to a change in g_*^s:

before: γ, e^+, e^- $\quad g_*^s = 2 + \frac{7}{8} \times 2 \times 2 = \frac{11}{2}$

after: γ $\quad (g_*^s)' = 2$

here we don't include ν's, since although they are relativistic they contribute equally before and after since their entropy is separately conserved.

Then we have

$$g_*^s T^3 = (g_*^s)' (T')^3$$

$$\Rightarrow \quad \frac{T'}{T} = \left(\frac{g_*^s}{g_*'^s}\right)^{1/3} = \left(\frac{11}{4}\right)^{1/3} \simeq 1.4$$

To be fully correct:

$$(g_*^s)_{tot} = g_*^s + \frac{7}{8} \times 6 \times \frac{4}{11}$$

$$(g_*^s)'_{tot} = (g_*^s)' + \frac{7}{8} \times 6 \times \frac{4}{11} \simeq 3.91$$

Now, since up to this $T = T_\gamma = T_\nu$, but after this the photons will have T' whereas neutrinos continue with T (both dropping as $1/a$)

therefore we have

$$\frac{T_\gamma}{T_\nu} \simeq 1.4 \quad \rightarrow \quad \boxed{T_\nu^{\,today} \simeq \frac{2.7\,^\circ K}{1.4} \simeq 1.9\,^\circ K}$$

after e^+e^- annihilation g_*^s does not change

· After e^+e^- annihilation, the next interesting epoch is when energy density in radiation equals matter energy density. Now as radiation we just count γ's and 3 generation of $\nu\bar{\nu}$, then

$$\rho_R = \frac{\pi^2}{30} g_* T^4 = \frac{\pi^2}{30} \underbrace{\left[2 + 3 \times 2 \times \frac{7}{8} \times \left(\frac{4}{11}\right)^{4/3}\right]}_{3.36} T^4$$

$$\Rightarrow \underset{\substack{\uparrow \\ T \sim \frac{1}{a} \sim (1+z)}}{\rho_R} = \frac{\pi^2}{30}\, 3.36\, (2.7^\circ K)^4 (1+z)^4$$

whereas for matter we have $\rho_M = \Omega_M h^2 \times 8 \times 10^{-47}\, GeV^4 (1+z)^3$

equality $\Rightarrow$ happens @

$$\boxed{1+z_{eq} = \Omega_M h^2\; 2.32 \times 10^4}$$

or

$$\boxed{T_{eq} = T_0 (1+z_{eq}) = 5.5\text{ eV}\; \Omega_M h^2}$$

$\simeq 3450$ for present cosmological parameters