Motivations for Inflation

Historically, motivation for inflation rests on the flatness problem, the horizon problem and the relic abundance problem. From a modern perspective though, the most important consequence of Inflation is the generation of density perturbations (which can be tested observationally) rather the solution to these problems. Let's go over them briefly:

1) Flatness problem

From the Friedmann equation $\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho$

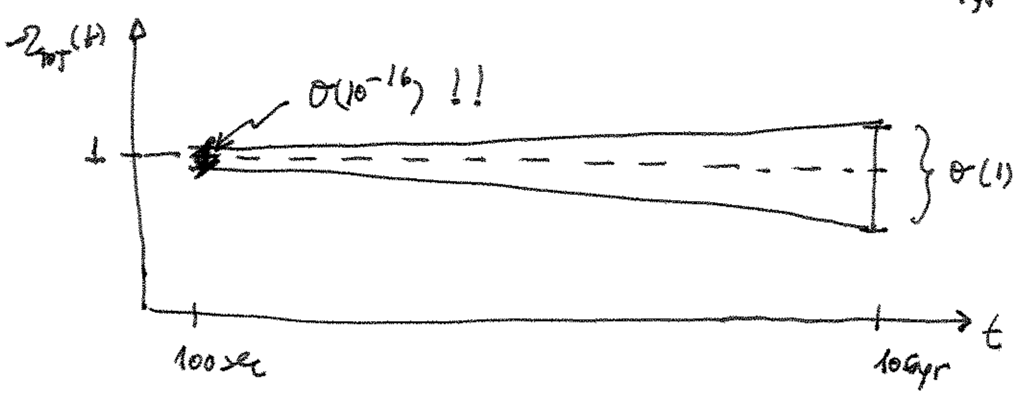
$$\Rightarrow \quad \Omega_{TOT} - 1 \equiv \frac{k}{a^2 H^2} = \frac{k}{\dot{a}^2} \sim \begin{cases} t & \text{RAD} \quad (a \sim t^{1/2}) \\ t^{2/3} & \text{MAT} \quad (a \sim t^{2/3}) \end{cases}$$

If $\Omega_{TOT} = 1 \Rightarrow k = 0$ and this stays so for ever.

However, if $\Omega_{TOT} \neq 1$, $|\Omega_{TOT} - 1|$ increases with time. For example, assumin RAD domination we have

$|\Omega_{TOT} - 1| \sim t$ whereas today $|\Omega_{TOT} - 1| \sim \mathcal{O}(1)$, then at BBN we must have:

$$|\Omega_{TOT} - 1|_{BBN} \lesssim \frac{100 \text{ sec}}{10 \text{ Gyr}} \underset{\substack{\uparrow \\ 1\text{yr} \sim \pi \times 10^7 \text{ sec}}}{\sim} 10^{-16}$$



Unless $\Omega_{TOT} = 1$, need an incredible fine tuning @ BBN, need to have $|\Omega_{TOT} - 1|$ be less than one part in 10^{16}. Otherwise it would be seen today as $\Omega_{TOT} > 1$ or $\Omega_{TOT} < 1$.

2) Horizon Problem

Why is the universe so smooth? When looking at the CMB we see that the temperature is uniform to a part in $\sim 10^5$, and there is no reason why this should be so. The CMB last interacted at decoupling when the horizon corresponds today to about 1 degree. There is no reason why points separated by more than 1 degree should have the same T. [If we consider earlier times than decoupling the problem becomes even worse]. Let's work it out. One expects uniform temperature on scales today that correspond to horizon scale @ decoupling:

$$\lambda_0^{dec} = \underbrace{\left(\frac{a_0}{a_{dec}}\right)}_{\substack{\text{scale} \\ \text{to today}}} \underbrace{d_H^{dec}}_{\text{horizon @ decoupling}} = \frac{a_0}{a_{dec}} \, a_{dec} \int_0^{t_{dec}} \frac{dt}{a(t)}$$

Since $t_{dec} \simeq 300{,}000\,yr \gg t_{EQ} \sim 5\times10^3 y$ we can assume MAT dominated in the integral, $\Rightarrow a(t) = a_{dec}\left(\frac{t}{t_{dec}}\right)^{2/3}$

$$\Rightarrow \lambda_0^{dec} \simeq \frac{a_0}{a_{dec}} t_{dec} \int_0^1 \frac{dx}{x^{2/3}} = \frac{a_0}{a_{dec}} 3 t_{dec}$$

How many of these volumes fit into horizon today? $d_H(t_0) = 3t_0$, then

$$\frac{d_H(t_0)}{\lambda_0^{dec}} = \frac{a_{dec}}{a_0} \frac{\cancel{3} t_0}{\cancel{3} t_{dec}} = \left(\frac{a_0}{a_{dec}}\right)^{1/2} \simeq \sqrt{z_{dec}} \overset{z_{dec}\sim10^3}{=} 33$$

Therefore, there are $(33)^3 \sim 40{,}000$ volumes inside our horizon that are causally disconnected @ decoupling. Why do they all have the same conditions?

The argument can be made even more extreme with BBN:

$$\lambda_0^{NUC} = \frac{a_0}{a_{NUC}} \times a_{NUC} \int_0^{t_{NUC}} \frac{dt}{a_{NUC}\sqrt{t/t_{NUC}}} = \frac{a_0}{a_{NUC}}\, 2\, t_{NUC}$$

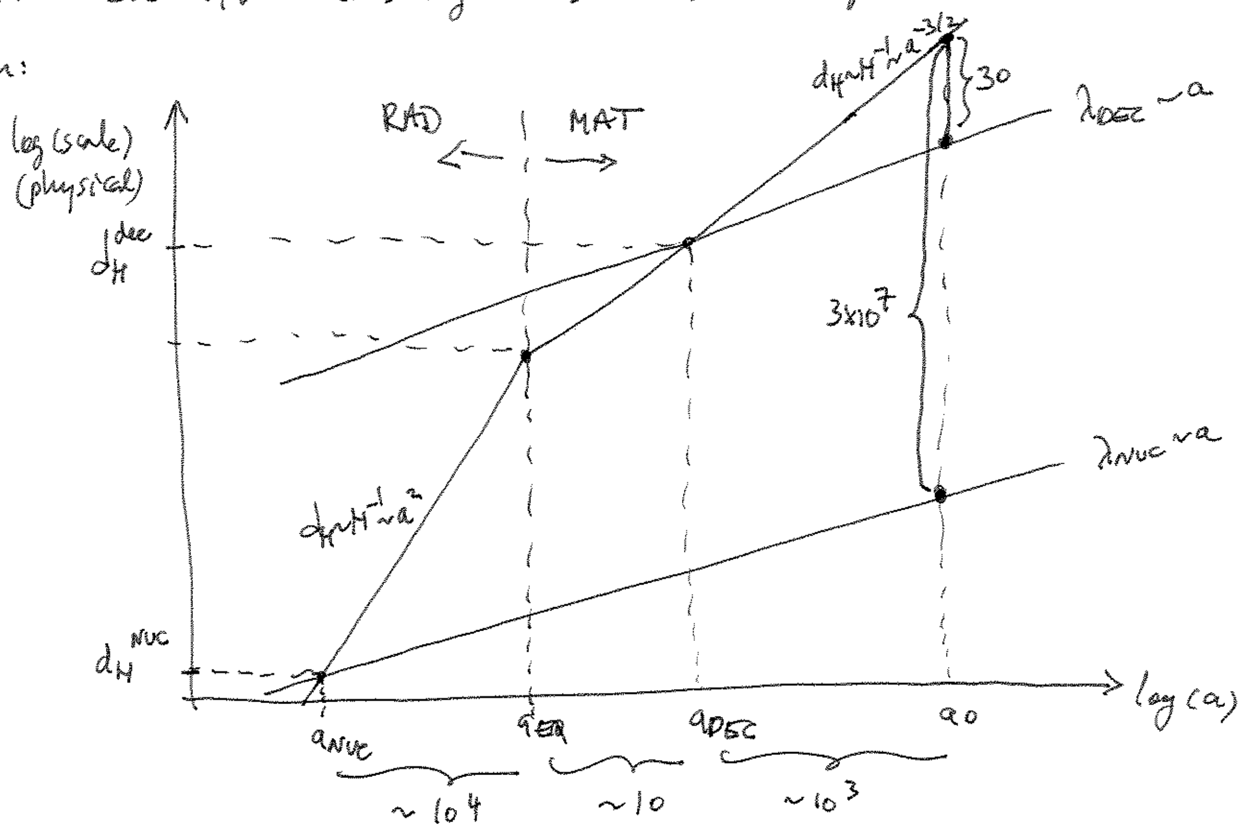
$$\Rightarrow \lambda_0^{NUC} = 2\, t_{NUC} \frac{a_0}{a_{EQ}} \frac{a_{EQ}}{a_{NUC}} \simeq 2\, t_{NUC} \times 10^4 \times \sqrt{\frac{t_{EQ}}{t_{NUC}}} \underset{\uparrow}{=} 2\times 10^8\, t_{NUC} \simeq 10^{-7} t_0$$

$t_{EQ} \simeq 3\times 10^3 y \sim 10^{10}$ sec

$t_{NUC} \simeq 100$ s $= \frac{1}{3} 10^{-15} t_0$

$d_H(t_0) = 3t_0 \Rightarrow (3\times 10^7)^3 = 10^{22}$ volumes in today H_0^{-1} radius are of size BBN horizon!

Remarkably, the synthesis of light elements appears to be the same (roughly) in all these $\sim 10^{22}$ causally disconnected independent volumes. The picture is then:



Recall that

$d_H = H^{-1} = 2t$ (RAD)

$d_H = 2H^{-1} = 3t$ (MAT)

- This brings another puzzle. Since perturbations in the CMB appear well in place already @ DEC, when where they created? Back in time those wavelengths are outside the horizon and there is no causal mechanism to physically create such perturbations.

3) <u>Unwanted Relics</u>

This is actually the original reason/motivation behind Guth's 1980 paper. In some grand unified theories (GUT) one produces a lot of heavy monopoles that survive to the present and would dominate the universe ($\Omega > 1$)

From a modern perspective, the dangerous relics are the gravitino (in supergravity, this is the spin 3/2 partner of the graviton) and also moduli (spin 0) from superstrings.

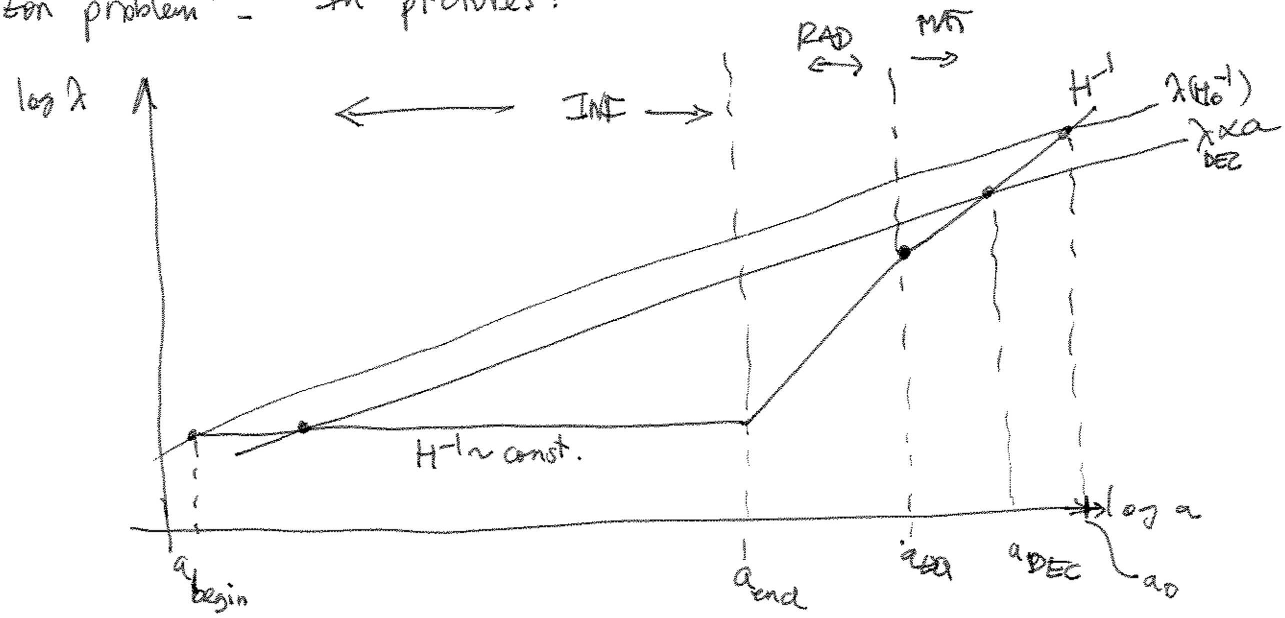
INFLATION

The horizon problem arises because during the evolution H^{-1} always scales faster than $\lambda \propto a$. If there is a long enough epoch in the past where that trend is reversed, we can have $\lambda < H^{-1}$ early enough. That's the idea.

Let's see: in STD cosmology $\frac{d}{dt}\left(\frac{H^{-1}}{a}\right) > 0$

to revert this we need $\frac{d}{dt}\left(\frac{H^{-1}}{a}\right) = -\frac{\ddot{a}}{\dot{a}^2} < 0 \Rightarrow \ddot{a} > 0$ (Inflation)

that is, an epoch of acceleration long enough in early universe can solve the horizon problem". In pictures:



Early enough all wavelengths were inside the Hubble radius during inflation.

Note here we assume exponential expansion, $H = $ const, during inflation. This is just a convenient choice, but any accelerated expansion will do, only need H^{-1} to scale <u>slower</u> than a.

- Once wavelengths are inside H^{-1} during inflation, causal physics can take place – Note however that once there is inflation you *must* make a distinction between H^{-1} and d_H since they are not behaving in the same fashion (like in RAD or MAT) -

For example, for exponential inflation, $a(t) \sim e^{Ht}$ $\quad H = \text{const.}$
$(\ddot{a} = H^2 a > 0)$

$\Rightarrow H^{-1} = \text{const.}$

but d_H grows exponentially,

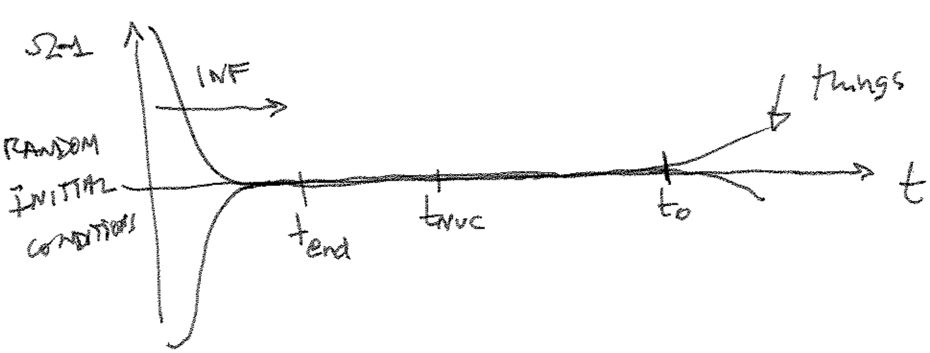
$$d_H(t) = a(t) \int_0^t \frac{dt}{a(t)} = e^{Ht} \int_0^t e^{-Ht}\, dt \sim H^{-1} e^{Ht}$$

say, take INF all the way to $t=0$

However, since $d_H > H^{-1}$ it is OK to talk about solving the "horizon problem" using H^{-1} instead of d_H - Also, remember that H^{-1} is what appears in equations of motion (being a local quantity), and therefore represents physically the distance over which things can typically move @ time t -

A period of accelerated expansion solves the other puzzles as well. It solves the flatness problem because

$$\Omega - 1 = \frac{k}{(aH)^2}$$ is now driven to zero very strongly during inflation



things may diverge later in the future, but if inflation lasts long enough this is *well* into the future

Similarly, a period of exponential (or quasi exponential) expansion drives to zero the abundance of unwanted relics, and for that matter, everything else. The trick is how to connect Inflation with RAD era, we'll discuss this shortly.

Finally, Inflation gives us a physical mechanism for creating density (and gravity waves) perturbations out of quantum mechanical fluctuations. This is by far the most interesting prediction of inflation because its details can be learned from observations and thus we can really probe into physics of the very early universe.

INFLATION AS SCALAR FIELD DYNAMICS

In order to have acceleration we need $(\rho + 3p) < 0$ for the effective equation of state. The simplest way to achieve ~~this~~ and have enough complexity to match into RAD era is with the dynamics of a scalar field ϕ rolling down some potential $V(\phi)$.

We assume a background FRW with Friedmann equation

$$H^2 = \frac{8\pi G}{3}\rho_\phi - \frac{k}{a^2} \underset{\substack{\text{EARLY UNIV.} \\ + \text{ INFLATION} \\ \text{drives curv} \to 0}}{\approx} \frac{8\pi G}{3}\rho_\phi$$

The energy density is going to be dominated by the inflaton ϕ, since all other forms will quickly redshift away as the universe expands exponentially, while ρ_ϕ stays approx. constant

The field ϕ will be taken to be homogeneous (we'll consider fluctuations later), therefore gradients will be neglected (with the caveat that if they were important, inflation may never start, though one can always think that where these gradients are small) is where we live ...)

A (minimally coupled) scalar field has lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - V(\phi) \underset{\phi \text{ homog.}}{\approx} \frac{1}{2}\dot\phi^2 - V(\phi)$$

the stress tensor corresponding to this is $T^{\mu\nu} = \partial^\mu\phi\,\partial^\nu\phi - \mathcal{L}g^{\mu\nu}$

$\underset{\phi \text{ homog.}}{\Rightarrow}$

$$\rho_\phi = \frac{\dot\phi^2}{2} + V(\phi) \quad \left\{ + \frac{|\nabla\phi|^2}{2a^2} \right\}$$

$$p_\phi = \frac{\dot\phi^2}{2} - V(\phi) \quad \left\{ - \frac{|\nabla\phi|^2}{6a^2} \right\}$$

from here we see that if grad's are important $p = -\frac{1}{3}\rho \Rightarrow \ddot a = 0 \Rightarrow$ no inflation. As long as inflation starts, grad's are diluted by a^{-2}.

If the potential energy dominates,

$$V(\phi) \gg \dot\phi^2/2$$

$\Rightarrow$ equation of state is $p_\phi \approx -\rho_\phi \Rightarrow$ exponential acceleration (like a cosmological constant)

The equation of motion for ϕ is obtained from stress-energy conservation $T^{\mu\nu}{}_{;\nu} = 0$ (or from the action, it's same)

$$\Rightarrow \quad \ddot\phi + \underbrace{3H\dot\phi}_{\text{friction due to expansion of the universe}} + V'(\phi) = 0$$

(or think as redshift of momentum $\dot\phi$)

Slow-roll inflation

In this popular model of inflation (which is all we will consider in this course) one has a very flat potential $V(\phi)$ and the field ϕ rolls down $V(\phi)$ very slowly, with terminal velocity and almost zero acceleration. In this regime,

$$\begin{cases} 3H\dot\phi + V'(\phi) \approx 0 \\ H^2 = \frac{8\pi G}{3}\rho_\phi \simeq \frac{8\pi G}{3}V(\phi) = \frac{V(\phi)}{3M_{pl}^2} \end{cases}$$

$$\left(M_{pl}^2 \equiv \frac{1}{8\pi G} \qquad m_{pl}^2 \equiv \frac{1}{G} \right)$$

This imposes some conditions of consistency on the shape of $V(\phi)$:

in order to neglect $\ddot{\phi}$ compared to, say, $3H\dot{\phi}$ we must have

$$\frac{\ddot{\phi}}{3H\dot{\phi}} \ll 1 \qquad \text{but from} \qquad \frac{d}{dt}\left(3H\dot{\phi} + V'(\phi)\right) \simeq 0 \Rightarrow 3\dot{H}\dot{\phi} + 3H\ddot{\phi} + V''\dot{\phi} \approx 0$$

$$\Rightarrow \quad \frac{\ddot{\phi}}{3H\dot{\phi}} \simeq -\frac{\dot{H}}{3H^2} - \frac{V''}{9H^2}$$

let's require each term be small (otherwise fine tuning, or cancellation, is required)

$$1 \gg \frac{V''}{9H^2} = \frac{V''}{9V} 3M_{pl}^2 = \frac{1}{3} M_{pl}^2 \frac{V''}{V} \equiv \frac{1}{3}\eta \quad \Rightarrow \quad \boxed{\eta \ll 1}$$

slow-roll parameters

Also: $2H\dot{H} = \dfrac{V'\dot{\phi}}{3M_{pl}^2} = -\dfrac{V'^2}{9M_{pl}^2 H}$

$$\Rightarrow -\frac{\dot{H}}{3H^2} = \frac{V'^2}{54 M_{pl}^2 H^4} = \frac{M_{pl}^2}{6}\frac{V'^2}{V^2} \equiv \frac{1}{3}\varepsilon \qquad \boxed{\varepsilon \ll 1}$$

Therefore, the two slow roll parameters $\quad \varepsilon = \dfrac{M_{pl}^2}{2}\left(\dfrac{V'}{V}\right)^2 \quad \eta = M_{pl}^2\left(\dfrac{V''}{V}\right)$

must be small, to be consistent.

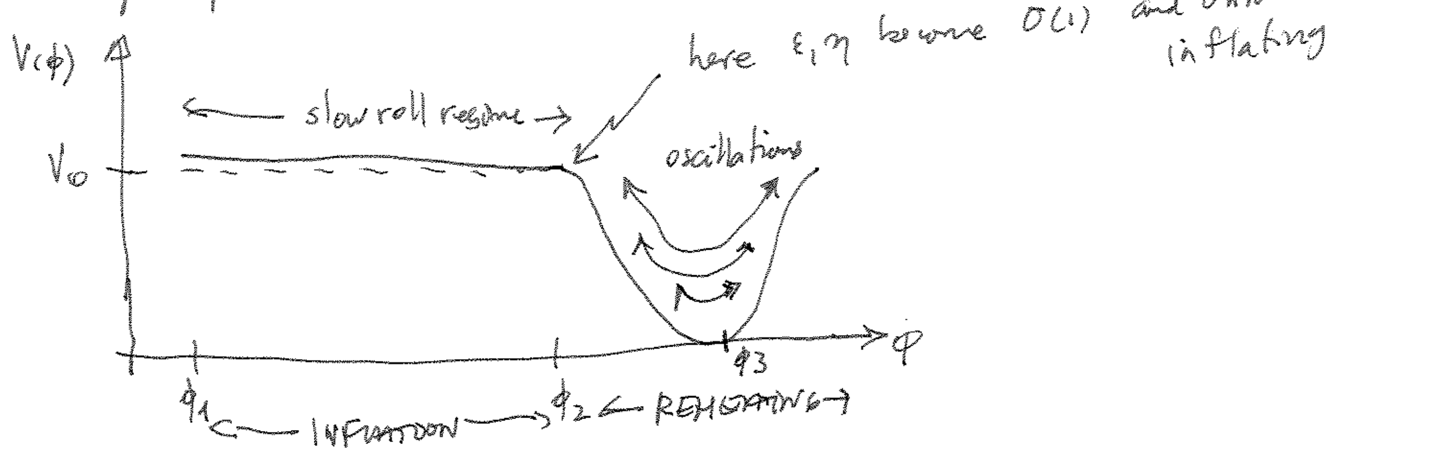
Note that these 2 conditions are <u>necessary</u> but <u>not sufficient</u>. They only restrict $V(\phi)$. Since the equations of motion are second order we can freely specify ϕ and $\dot{\phi}$ as initial conditions. At some given ϕ we may have $\varepsilon, \eta \ll 1$, but if we give $\dot{\phi}$ arbitrarily, we can violate the desired $3H\dot{\phi} \simeq V'$. Therefore we need both $\varepsilon, \eta \ll 1$ <u>and</u> $3H\dot{\phi} \simeq -V'(\phi)$. With these you can show that $\frac{\dot{\phi}^2}{2} \ll V(\phi)$

Reheating

Inflation must stop at some point, and the potential energy in the inflaton field must be converted into radiation degrees of freedom, eventually matching to the evolution in the RD era, after which we have BBN, et

There are two main mechanisms by which this conversion takes place. For ϕ coupled to fermions (i.e. decaying into fermions) the process is slow and can be parametrized phenomenologically in a very simple way, this is what we will describe. For decay into bosons is more complicated, the decay is fast an proceeds through parametric resonance.
↳ compared to H^{-1}

The simple picture is as follows:



Once inflation stops when V becomes steep, we enter a regime where ϕ oscillates around ϕ_3 with frequency $\omega^2 \sim V''(\phi_3)$. At the same time ϕ is coupled to a bath with decay rate ~~constant~~ $\Gamma_\phi \ll H$, so since $\Gamma_\phi^{-1} \gg H^{-1}$ it takes many oscillations until it drains all the energy out of ϕ, the oscillations being slowly damped.

The effective equation of motion for ϕ in this regime can be written as

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma_\phi \dot{\phi} + V'(\phi) = 0$$

Now, since $\rho_\phi = \frac{\dot{\phi}^2}{2} + V \quad \Rightarrow \quad \dot{\rho}_\phi = \dot{\phi}\ddot{\phi} + V'\dot{\phi}$

$$\Rightarrow \quad \dot{\rho}_\phi + (3H + \Gamma_\phi)\dot{\phi}^2 = 0$$

Now, doing a time average over oscillations, we know for a harmonic potential that: $\langle \frac{\dot\phi^2}{2} \rangle = \langle V \rangle = \frac{1}{2}\bar\rho_\phi$ $\qquad \bar\rho_\phi \equiv \langle \rho_\phi \rangle$

$$\Rightarrow \langle \dot\phi^2 \rangle \simeq \bar\rho_\phi \quad \Rightarrow \quad \boxed{\dot{\bar\rho}_\phi + (3H + \Gamma_\phi)\,\bar\rho_\phi = 0}$$

since $\Gamma_\phi \ll H \Rightarrow \dot{\bar\rho}_\phi + 3H\bar\rho_\phi \simeq 0$

$\Rightarrow \bar\rho_\phi \sim \frac{1}{a^3}$ like non-relativistic matter!

eventually, as H goes down, $\Gamma_\phi \sim H$ and all energy is converted into radiation degrees of freedom. The reheating temperature T_{RH} obeys:

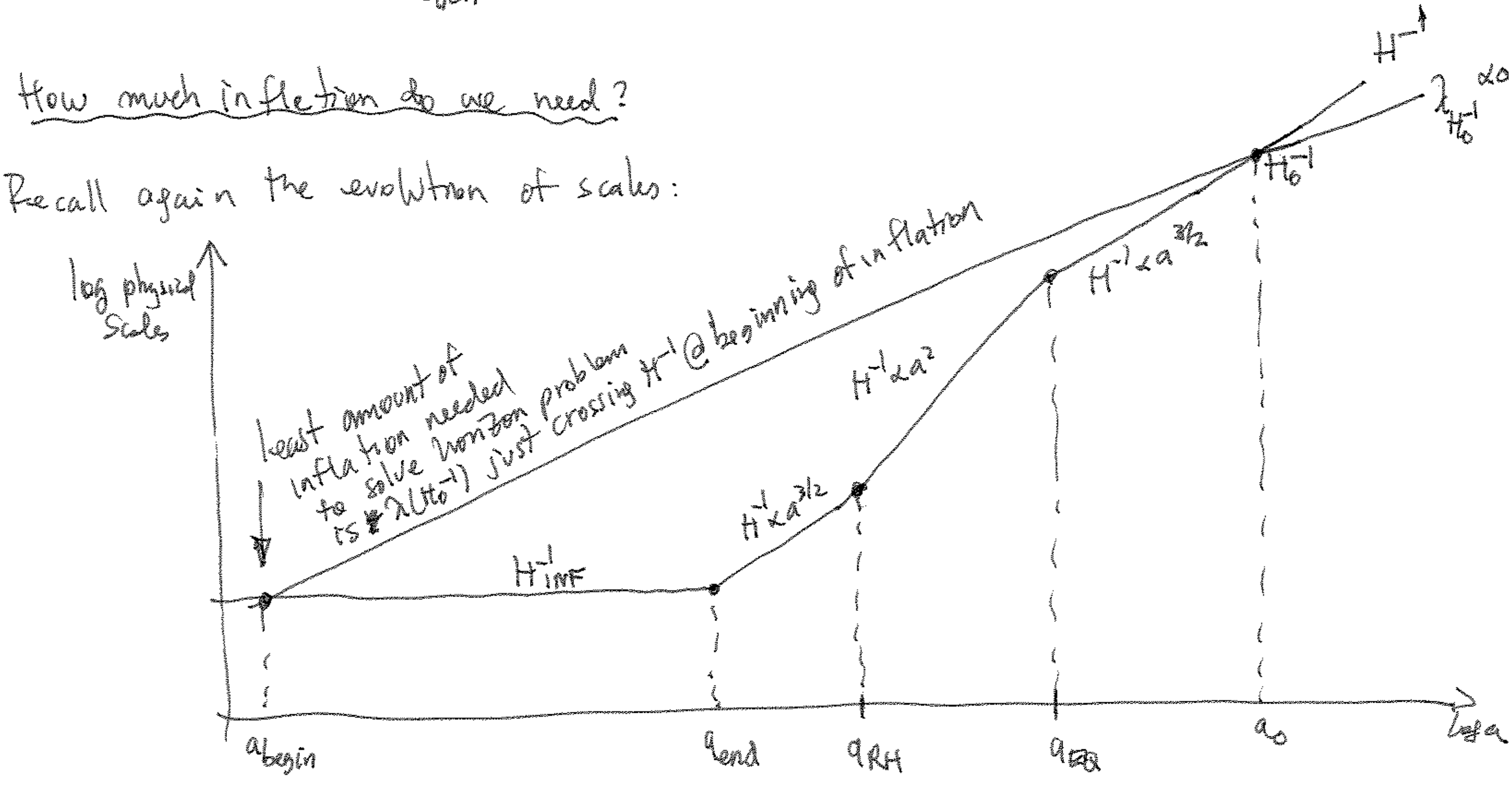
$$V_\phi \underbrace{\left(\frac{a_{end}}{a_{RH}}\right)^3}_{\substack{\text{redshift factor} \\ \text{from end of INF} \\ \text{to end of REHEAT}}} = \rho_R = \frac{g_* \pi^2}{30} T_{RH}^4$$

radiation energy density for thermalized decay products

$$\Rightarrow V_0 \left(\frac{a_{end}}{a_{RH}}\right)^3 \sim T_{RH}^4$$

How much inflation do we need?

Recall again the evolution of scales:



the minimal requirement is to ask that the scale corresponding to the Hubble radius today was just inside H^{-1} during inflation, then

$$\underbrace{\lambda_{H_0^{-1}}}_{3000\,h^{-1}\,\mathrm{Mpc}} \simeq H^{-1}_{INF} \underbrace{\left(\frac{a_0}{a_{begin}}\right)}_{\text{strech factor}} = \sqrt{\frac{3M_{pl}^2}{V_0}} \underbrace{\left(\frac{a_{end}}{a_{begin}}\right)}_{e^{N_{min}}} \underbrace{\left(\frac{a_{RH}}{a_{end}}\right)}_{\left(\frac{V_0}{T_{RH}^4}\right)^{1/3}} \underbrace{\left(\frac{a_0}{a_{RH}}\right)}_{\frac{T_{RH}}{T_0}}$$

assuming S is conserved, i.e. $g_*^S = $ const.

then the minimum number of e-folds N_{min} of inflation is:

$$N_{min} \simeq 53 + \frac{2}{3} \ln\left(\frac{V_0^{1/4}}{10^{14}\,\mathrm{GeV}}\right) + \frac{1}{3} \ln\left(\frac{T_{RH}}{10^{10}\,\mathrm{GeV}}\right)$$

$(N_{TOT} \geq N_{min})$ for $V_0^{1/4}, T_{RH}$ from 1 GeV to 10^{18} GeV N_{min} : 24 $\rightarrow$ 68

so this is somewhat model dependent, of course.