Homework Set #1 (Due 10/5 in Class)

1. Making reasonable assumptions, *estimate* the following,

a) today's energy density of radiation (in terms of the critical density), Ω_R^0.

b) the number of supernovae that explode per unit time in the observable universe.

c) how much has the universe expanded since its energy density was equal to the Planck energy. Express the result in terms of the e-folds N of expansion, where $a = \exp N$.

d) how far away was a galaxy when its light was emitted assuming it has a redshift of $z = 0.1$.

e) the time it takes for a new galaxy to come within our horizon.

2. Show that the Cosmological Principle (CP) plus Galilean invariance leads to Hubble's law.
Hint: Consider 2 comoving observers that see a galaxy also at fixed comoving position. Demand addition of velocities plus the CP, and infer that the relation between velocity $\mathbf{v}$ and distance $\mathbf{r}$ must be linear. Then show using symmetry arguments that the linear relation between the two vectors must be through a unit matrix, i.e. $\mathbf{v} = H\,\mathbf{r}$.

3. Olber's Paradox: Why is the night sky dark?
Consider light sources (stars or galaxies) with mean number density n_s and luminosity L_s (energy per unit time) independent of position, as demanded by the Cosmological Principle.

a) Show that in an Euclidean static universe where sources have an infinite lifetime the mean radiation energy density diverges.

b) Relax the assumption of static universe, use the FRW metric and assume that sources turn on at some characteristic time t_s. Assuming that the *comoving* number density of sources is constant, give an expression for the mean radiation energy density.

c) Assuming a flat matter dominated universe, compare the expanding case to the static case. How important is the supression of background light due to cosmological expansion as a function of t_s/t_0? (t_0 is the age of the universe).

4. In class we derived the relation $d_L = r_1 a_0(1+z)$ for the *luminosity distance* d_L as a function of redshift z, present scale factor a_0 and comoving distance to the source r_1.

a) Show by expanding about recent times that this expression for d_L can be rewritten as a small z expansion,

$$H_0 d_L = z + \frac{1}{2}(1 - q_0)\ z^2 + \ldots, \tag{1}$$

where H_0 and q_0 are the present values of the Hubble constant and the deceleration parameter, respectively. *Hint:* Recall that $\int_0^{r_1} dr/\sqrt{1-kr^2} = \int_{t_1}^{t_0} dt/a(t)$.

b) Calculate the cubic term in z as well and show that it depends on curvature (for the flat case you should obtain the result presented in class).

This relation plays a central role in determining the acceleration of the universe by measuring the flux from distant supernovae type Ia as a function of redshift.

5. The *Chandrasekhar mass limit* for white dwarfs (stars supported against gravity by degeneracy pressure of relativistic electrons) is the fundamental result that makes supernovae type Ia standarizable candles.

Derive this mass limit as follows. Assume the star is made out of electrons and protons.

a) Use the uncertainty principle plus the exclusion principle to estimate that the Fermi energy (dominated by electrons) per fermion is

$$E_F \sim \frac{\hbar c N^{1/3}}{R}, \tag{2}$$

where N is the number of fermions (electrons and protons) in the star and R its radius.

b) Add to this the gravitational potential energy per fermion to estimate the total energy E. Note that even though the pressure is dominated by the degenerate relativistic electrons, the mass is dominated by protons (with mass m_P).

c) Argue that for large N the total energy $E < 0$ and there is no stable equilibrium, leading to collapse. From this argument derive the maximum mass

$$M_{\text{Chandra}} \sim m_{\text{P}} \left(\frac{\hbar c}{G m_{\text{P}}^2} \right)^{3/2}. \tag{3}$$

Show that this corresponds to about $1.5 M_\odot$.