Homework Set #3 (Due 11/16 in Class)

1. The "horizon problem".

a) Show that in the standard Big Bang model the horizon distance today is to a very good approximation $d_H \approx 2H^{-1}$, even though the universe changed from radiation to matter dominated sometime in the past. Use reasonable values for t_{eq}/t_0. At what redshift does this approximation become better than 10%?

b) Consider now the evolution of d_H and H^{-1} when there is a period of inflation. Assume the universe is radiation dominated before and after the period of inflation, which lasts 60 e-folds. Show that the horizon and the Hubble radius are very different today. Explain why the behavior of d_H solves the "horizon problem". Sketch a graph of scales, H^{-1}, and d_H as a function of scale factor.

c) Show that once scales are in causal contact (as measured by d_H) *they stay forever* in causal contact.

d) The explanation in b) is not the usual explanation found in discussions about inflation (e.g. in books), which actually involves H^{-1} rather than d_H. Why is this so, and why does the "standard" explanation make sense?

2. Gaussian Random Fields and Vacuum Fluctuations

a) Show that for a Gaussian probability distribution function (PDF), $P_G(\phi) \equiv (2\pi\sigma^2)^{-1/2} \exp[-\phi^2/(2\sigma^2)]$, the moments are given by $\langle\, \phi^{2n} \,\rangle = (2n-1)!!\, \langle\, \phi^2 \,\rangle^n$, where $\langle\, \phi^2 \,\rangle = \sigma^2$ and $(2n-1)!! = (2n-1)(2n-3)\ldots 1$.

b) All the moments can be generated from the *moment generating function* $M(\lambda) \equiv \langle\, \exp(\lambda\phi) \,\rangle$, i.e. $\langle\, \phi^n \,\rangle = (d^n M(\lambda)/d\lambda^n)_{\lambda=0}$. Using the result in a), show that for a Gaussian field $M(\lambda) = \exp[\lambda^2\sigma^2/2]$.

c) Show that vacuum quantum fluctuations obey Gaussian statistics.
Hint: Expand each mode $\phi_k = w_k\, a_k + w_k^*\, a_k^\dagger$ in terms of creation and annihilation operators $a_k^\dagger$ and a_k, obeying the usual commutation relations and calculate the moment generating function for such a mode. For operators A and B that commute with their commutator $[A, B]$, the following may come handy: $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$.

3. Fluctuations from Inflation.

a) Consider the model of chaotic inflation, where the potential is given by $V(\phi) = V_0\phi^p$, with $p \geq 2$. Calculate the slow-roll parameters $\epsilon(\phi)$ and $\eta(\phi)$ in terms of parameters of the potential and find the amplitude of the field when inflation ends (justify your prescription here) and 50 e-folds before the end of inflation.

b) Assuming that the amplitude of fluctuations for modes that become larger than H^{-1} 50 e-folds before the end of inflation is $\Delta(k) \sim 10^{-11}$ (as observed by the COBE and WMAP satellites), calculate the energy scale of inflation, the scalar spectral index n_S and the tensor spectral index n_T. How important is the contribution of tensor modes compared to scalar modes when $p = 4$?

c) Assuming the slow-roll approximation, find the most general form of the inflaton potential so that scalar perturbations are exactly scale-invariant.