Homework Set #4 (Due 12/7)

1. The linear evolution of the dark matter power spectrum $P(k,\tau)$ after decoupling is given by

$$P(k,\tau) = [D(\tau)]^2\ [T(k)]^2\ P_p(k), \tag{1}$$

where $D(\tau)$ is the linear growth factor [i.e. $D(\tau_{dec}) \equiv 1$], $T(k)$ is the transfer function and $P_p(k)$ is the primordial power spectrum, e.g. $P_p(k) = (4/25)(k/\mathcal{H})^4 P_{\mathcal{R}}(k)$ where $P_{\mathcal{R}}(k) \sim k^{n_s-4}$ is the curvature perturbation power spectrum generated by inflation, with n_s the scalar spectral index ($n_s = 1$ for a Harrison-Zel'dovich spectrum).

a) From Eq. (1) it follows that the present value of the spectral index, $n(k) = d\ln P(k)/d\ln k$, is affected by the physics of inflation, and the dark matter and radiation content of the universe. Does the spectral index (at some fiducial scale) increase or decrease when,

   i) The curvature of the inflationary potential is increased.
   ii) The amount of dark matter is increased.
   iii) The amount of radiation is increased.

   In each case explain the physics of why the spectral index changes.

b) Although both induce changes in the spectral index, the inflationary effect on the power spectrum shape is somewhat different from that due to the energy contents. Explain what is this difference (assume that the overall normalization of the spectrum is fixed by comparison with observations).

c) The transfer function will be exponentially cutoff for scales $k > k_{FS}$ due to free streaming. Therefore, in the linear approximation, the maximum value of $\Delta(k) = 4\pi k^3 P(k)$ will be around $k \sim k_{FS}$. Estimate the mass of the dark matter particle so that the first objects to form are of galactic mass, $M \sim 10^{10} M_\odot$.

2.

a) Consider the evolution of baryon perturbations right after decoupling. Assume the universe is matter dominated, and consists only of baryons and dark matter, with $\delta_B(a_{dec}) = 0$, and dark matter fluctuations in their growing mode by decoupling time. You may assume the gravity is dominated by the dark matter perturbations alone even after decoupling, and that the baryons have a sound speed $c_s^2 = (5/3)kT/m_B \propto 1/a$. Write down the equations of motion for both components and find the general solution.

b) Show that baryons trace the dark matter soon after recombination at scales larger than the Jeans length. What is the relationship between baryons and dark matter at small scales?

c) Suposse there is a component in the universe (e.g. galaxies) that forms at some characteristic time $t = t_*$, with density contrast $\delta_g = b_0\delta$ at $t = t_*$, where δ is the dark matter density. Assuming that the number of these objects is conserved, and that their velocities equal those of dark matter particles, show that the "bias factor" $b \equiv \delta_g/\delta$ evolves towards unity at late times.