We now discuss functions of a complex variable, only touching on very simple tools we need later for Fourier analysis. Keep in mind this is by no means a proper discussion of the basics of complex variable techniques (that would take many weeks of class in itself).

**Complex Numbers**

A general complex number is written as

\[ z = x + iy \]

where \( x \) and \( y \) are real numbers, and \( i = \sqrt{-1} \) (known as the imaginary number). \( x \) is said to be the real part of \( z \), and \( y \) the imaginary part of \( z \):

\[ x = \text{Re}(z) \]
\[ y = \text{Im}(z) \]

It is often convenient to use a graphical representation of complex variables. The so-called complex plane, a 2-dimensional space where horizontal axis is real part and vertical axis is imaginary part:

\[ |z| = \sqrt{x^2 + y^2} \quad \text{absolute value of a complex number} \]

\[ \bar{z} = x - iy \quad \text{is complex conjugate} \]

(\( \bar{z} \) inverts imaginary part)

So: \( |z|^2 = z \bar{z} \)

One can also represent complex numbers in polar form, using magnitudes and angles. To do so, we take advantage of

\[ e^{i\theta} = \cos \theta + i \sin \theta \]

This can be understood by Taylor series of the exponential:  
\[ e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \]

\[ e^{ix} = \left( 1 - \frac{x^2}{2} + \frac{x^4}{4} + \cdots \right) + i \left( \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \right) = \cos \theta + i \sin \theta \]
In polar form we can write

\[ z = r \, \text{e}^{i \theta} = \text{Re} z + i \, \text{Im} z \quad \text{(r, \theta \text{ are real numbers})} \]

\[ |z| = |r \, \text{e}^{i \theta}| = r \, |\text{e}^{i \theta}| \quad \text{but} \quad |\text{e}^{i \theta}| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1 \]

\[ |z| = r \]

complex conjugation gives \( z^* = r \, \text{e}^{-i \theta} \)

From above we get \( \tan \theta = \frac{y}{x} \quad \Rightarrow \quad \theta = \arctan \frac{y}{x} \)

\( \theta \) is called the argument (or phase) of \( z \) : \( \theta = \arg z \)

Functions of complex variable

A function \( w \) of complex variable \( z \), takes a complex number and gives another (just like function of real numbers return real numbers) - We can write

\[ w = w(z) = u(x, y) + i \, v(x, y) \quad \text{where} \]

\[ \begin{align*}
  u & = \text{Re} w \\
  v & = \text{Im} w 
\end{align*} \]

\( z = x + iy \)

\( u(x, y) \) and \( v(x, y) \) are real functions of real numbers \( x \) & \( y \) -

e.g. \( w(z) = z^2 \quad \Rightarrow \quad w(z) = (x + iy)^2 = x^2 - y^2 + i \, 2xy \)

\[ \Rightarrow u(x, y) = x^2 - y^2 \quad v(x, y) = 2xy \]

One can think of functions of complex variable as mapping the \( z \) complex plane into the \( w \) complex plane:

Some elementary functions:

\[ \sin z = \sin (x + iy) \]

\[ \cos z = \cos (x + iy) \]
What do we mean by $\sin z$?

You can either think of $\sin z$ using Taylor series, e.g.

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

or use the exponential notation,

$$e^{iz} = \cos z + i \sin z$$

$$\Rightarrow \sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

So, for example:

$$\sin (x+iy) = \frac{e^{ix} - e^{-ix}}{2i} = \frac{1}{2} \left( e^y e^{ix} - e^{-y} e^{-ix} \right)$$

$$= \frac{-i}{2} \left[ \cos x \left( e^y - e^{-y} \right) + i \sin x \left( e^y + e^{-y} \right) \right] = \sin x \cosh y + i \cos x \sinh y$$

Another interesting case is the logarithm,

$$\ln z = \ln (re^{i\theta}) = \ln r + \ln e^{i\theta} = \ln r + i\theta$$

So one could also write,

$$\ln z = \ln \sqrt{x^2+y^2} + i \tan^{-1} (\frac{y}{x})$$

E.g. $z = 2 e^{i\pi/4} \Rightarrow \ln z = \ln 2 + i\pi/4$

\[2\sqrt{2} + i \frac{\sqrt{2}}{2}\]

Derivatives: Cauchy-Riemann conditions

Let's take a look at derivatives of complex functions, defined in the usual way

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \frac{df}{dz} = f'(z)$$

value $f'(z)$ is a complex function of variable $z$.

This will make sense only if the derivative does not depend on how we take the limit $\delta z \to 0$, we can go with $\delta z$ real or
δz imaginary and should give the same answer (this is a very restrictive condition). Pictorially, for a derivative at \( z = z_0 \)

If we write

\[ \delta z = \delta x + i \delta y \]

and

\[ \delta f = \delta u + i \delta v \]

\[ \Rightarrow \frac{\delta f}{\delta z} = \lim_{\delta z \to 0} \frac{\delta u + i \delta v}{\delta x + i \delta y} \]

Let's first take \( \delta y \to 0 \) \( \delta x \to 0 \) 1. Then

\[ \frac{df}{dz} \bigg|_0 = \lim_{\delta x \to 0} \frac{\delta u + i \delta v}{\delta x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \]

In the second approach \( \delta y \to 0 \), \( \delta x \to 0 \)

\[ \frac{df}{dz} \bigg|_0 = \lim_{\delta y \to 0} \frac{\delta u + i \delta v}{i \delta y} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \]

If we are to have a derivative \( \frac{df}{dz} \) independent of approach, we need these two operations to give the same complex number:

\[ \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \]

\[ \Rightarrow \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \]

These are known as Cauchy-Riemann conditions. These are necessary conditions for the derivative to exist. Sufficient conditions require the partial derivatives of \( u, v \) to exist and be continuous in the neighborhood of a point \( z_0 \). Such functions are called analytic @ \( z_0 \) (i.e., when a function is differentiable @ \( z_0 \) and neighborhood around it).

Example:

\[ f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + i 2xy \]

\[ u = x^2 - y^2 \]
\[ v = 2xy \]

\[ \frac{\partial u}{\partial x} = 2x \quad \frac{\partial u}{\partial y} = -2y \]
\[ \frac{\partial v}{\partial x} = 2x \quad \frac{\partial v}{\partial y} = 2y \]

Satisfies C-R everywhere, and they are continuous => analytic everywhere.
\[
\frac{df}{dz} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = 2x + i 2y = 2(x + iy) = 2z
\]

Using (i), you can write also (ii) same!

Note that then, \( \frac{df}{dz} = 2z \), as expected!

\[
f(x) = x^2 \implies u = x, v = -y
\]

\[
\frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = -1 \implies \text{C-R does not work!} \implies f(x) = x^2 \text{ is not analytic}
\]

(though it is continuous everywhere!)

Note: the same rules for products, chain rule, ratios of variables still hold for derivation of analytic function, e.g.

\[
\frac{d}{dz} (fg) = f g' + f' g
\]

\[
\frac{1}{dz} e^{f(z)} = e^{f(z)} f' \quad \text{etc.}
\]

Example: find the analytic function \( w(z) = u(x, y) + i v(x, y) \)

\[
\text{if} \quad v(x, y) = e^{-y} \sin x
\]

Use C-R to find \( u \):

\[
\begin{align*}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} = -e^{-y} \sin x \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} = -e^{-y} \cos x
\end{align*}
\]

\[
(1) \quad u = -\int e^{-y} \sin x \, dx + Ay = \int e^{-y} \cos x + Ay
\]

\[
(2) \quad u = -\int e^{-y} \cos x \, dy + Bx = \int e^{-y} \cos x + Bx
\]

Matching both, at most \( Ay = Bx = \text{const.} \)

\[
\Rightarrow \quad u = e^{-y} \cos x + c
\]

Let's take \( c = 0 \) \( \Rightarrow \quad w = e^{-y} \left( \cos x + i \sin y \right) \)
Another example: \( u = 6x + 3y \) find \( v \) so that \( w = u + iv \) is analytic.

\[
\begin{align*}
\frac{\partial u}{\partial x} &= 6 = \frac{\partial v}{\partial y} & \Rightarrow v &= 6y + \Phi(x) \\
\frac{\partial u}{\partial y} &= 3 = -\frac{\partial v}{\partial x} & \Rightarrow v &= -3x + \Psi(y)
\end{align*}
\]

\( \Phi(x) = -3x \)
\( \Psi(y) = 6y \)

\( \Rightarrow v = 6y - 3x \) \( \Rightarrow w = 6x + 3y + i(6y - 3x) \)

\( \Rightarrow w(z) = 6z - 3iz = (6-3i)z \)

**Cauchy Theorem & Formula**

An important consequence of analytic functions is Cauchy's theorem.

Let's calculate

\[
I = \oint_C dz w(z)
\]

where \( C \) is some contour in complex plane and \( w(z) \) is analytic everywhere inside \( C \).

We can write \( w = u + iv \) \( dz = dx + idy \)

\( \Rightarrow I = \oint_C (dx + idy)(u + iv) = \oint_C (u \, dx - v \, dy) + i \oint_C (u \, dy + v \, dx) \)

Now we can use Stokes theorem

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}
\]

For first integral we let

\( V_x = u \quad V_y = -v \) \( \Rightarrow \oint_C (u \, dx - v \, dy) = \iint_S d\mathbf{S} \left( \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) = \int_0^1 \int_0^1 \text{from C-R condition} \)

For second integral we use \( V_x = v \quad V_y = u \)

\( \Rightarrow \oint_C (u \, dy + v \, dx) = \iint_S d\mathbf{S} \left( \frac{\partial V_x}{\partial y} - \frac{\partial V_y}{\partial x} \right) = 0 \quad \text{C-R condition} \)

Then we arrive at Cauchy's theorem. An analytic function has zero integral in complex plane,

\[
\oint_C dz w(z) = 0 \quad \text{if \( w(z) \) is analytic inside \( C \).}
\]

It's interesting to note that because integrating over regions where a function is analytic does not contribute to the value of the integral,
One can add such regions outside for making calculations easier. This trick is known as contour deformation, e.g.

\[
\int_{\text{Im} z} \Rightarrow \text{closed contour } \quad \text{where } w \text{ is analytic}
\]

\[
\int_{c_0} w(z) \, dz = \int_{c_0} w(z) \, dz + \oint w(z) \, dz = \int_{c_0'} w(z) \, dz
\]

where integrals along "straight lines" in opposite directions cancel. So, integral along contour \(c_0\) can be "deformed" to be made along \(c_0'\), as long as \(w(z)\) is analytic inside \(c_0\).

An important consequence of Cauchy's theorem is Cauchy's integral formula. Consider the integral,

\[
I = \oint_{c_0} \frac{f(z)}{z-z_0} \quad \text{where } f(z) \text{ is analytic inside } c_0
\]

if \(z_0\) is not inside \(c_0\) then \(\frac{1}{z-z_0}\) is analytic, thus \(\frac{f(z)}{z-z_0}\) is analytic inside \(c_0\) and \(I=0\). But what about if \(z_0\) is inside \(c_0\)?

\(\frac{1}{z-z_0}\) is analytic everywhere except at \(z=z_0\), where the partial derivatives in the C-R conditions diverge. So, if \(z_0\) is inside \(c_0\), \(\frac{f(z)}{z-z_0}\) is not analytic and we cannot invoke Cauchy's theorem to say \(I=0\).

Note that the integral is along a contour that encloses \(z_0\), but doesn't go through \(z_0\), so as \(\infty\) as we integrate, true integral is well defined.

Now, because integrand is analytic everywhere except at \(z=z_0\), we can deform the contour down to an infinitesimal circle around \(z=z_0\), of radius \(\varepsilon\)

\[
\int_{\gamma} \Rightarrow \gamma = z_0 = \varepsilon e^{i\theta} \quad \varepsilon \to \infty
\]

\[
dz = \varepsilon e^{i\theta} \, d\theta
\]
\[ i = \int_{C} \frac{f(z_0 + \varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} \, \varepsilon e^{i\theta} \to i f(z_0), \quad \int_{C} f(z) \, dz = 2\pi i f(z_0) \text{ if inside } C \quad \text{(Cauchy integral formula)} \]

Remark: Note this is a remarkable result: it says that if you specify the values of \( f(z) \) at a contour \( C \), you have determined \( f(z_0) \) inside! (This is similar to Gauss' law.)

We can obtain a more general formula by taking derivatives with respect to \( z \) on both sides:

\[ \text{Res} \frac{d}{dz} \left( \frac{f(z)}{z - z_0} \right) = \frac{f^{(n)}(z_0)}{n!} - \frac{1}{2\pi i} \int_{C} \frac{f(z)}{(z - z_0)^{n+1}} \, dz \]

In general:

\[ \text{Res} \frac{d}{dz} \left( \frac{f(z)}{(z - z_0)^n} \right) = \frac{f^{(n)}(z_0)}{n!} - \frac{1}{2\pi i} \int_{C} \frac{f(z)}{(z - z_0)^{n+1}} \, dz \]

where \( f \) is analytic within \( C \) and even more remarkably:

if \( f \) is known in \( C \) we know all the derivatives inside \( C \).

Examples:

a) \[ \int_{C} \frac{dz}{z-1} \]

\( f(z) = 1 \) \quad \( z_0 = 1 \) \quad \text{is inside } C \Rightarrow \int_{C} \frac{dz}{z-1} = 2\pi i \]

b) \[ \int_{C} \frac{dz}{(z-1)^2} \]

in same circle

Now

\[ \int_{C} \frac{dz}{(z-1)^2} = \sum_{n=0}^{\infty} \frac{2\pi i}{n!} \frac{d^n f}{dz^n} \bigg|_{z=1} \]

\( n=1 \)

\( f = 1 \)

Now, how about \[ \int_{C} \frac{dz}{z^2 - 1} \]

along same circle?
We decompose
\[
\frac{1}{z^2-1} = \frac{1}{2} \left( \frac{1}{z-1} - \frac{1}{z+1} \right)
\]

\[
\int_0^{2\pi} \frac{dz}{z^2-1} = \frac{1}{2} \left[ \int_0^{2\pi} \frac{dz}{z-1} + \int_0^{2\pi} \frac{dz}{z+1} \right] = 0
\]

\[
2\pi i \times f(-1) = 2\pi i
\]

**Other important consequences of analytic functions**

If \( f(z) = u + iv \) is analytic, we can derive a couple of interesting properties. First, let's compute the 2D Laplacian of \( u \) and \( v \):

\[
\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = 0
\]

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\]

and also

\[
\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0
\]

So both \( u \) and \( v \) are harmonic functions \( \nabla^2 u = \nabla^2 v = 0 \).

Also:

\[
\nabla u \cdot \nabla v = \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} = 0
\]

so \( \nabla u \perp \nabla v \) or \( u = \text{const.} \) (equipotentials of \( u \))

are perpendicular to \( v = \text{const.} \) (equipotentials of \( v \)).