The Dirac Delta Function

In vector and matrix algebra one has the identity matrix,

\[ \mathbf{1} \cdot \mathbf{v} = \mathbf{v} \]

\[ (1) \text{} \quad (\mathbf{1})_{ij} = \delta_{ij} \quad \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \]

In components,

\[ \sum_j (\mathbf{1})_{ij} \mathbf{v}_j = \mathbf{v}_i \]

The idea is to generalize this concept to continuous functions, rather than discrete indices, \( i, j \). We look for a "function" \( \delta \) such that

\[ \int_{\mathbb{R}} \delta(x_0, x) f(x) \, dx = f(x_0) \]

replaces \( \sum_j "i" \cdot "j" \cdot \mathbf{v}_j \)

A naive approach would be to take

\[ I(x_0, x) = \begin{cases} 1 & x = x_0 \\ 0 & x \neq x_0 \end{cases} \]

but this doesn't work, because the integral will be zero except at \( x = x_0 \) (where it is finite), but such an integral is zero, because a single point with finite value gives a vanishing contribution to an integral.

The only way to make such a thing pick up the value of \( f \) at \( x = x_0 \), is to make

\[ I(x_0, x_0) = \infty \]

so we can get a limit contribution to the integral. From this it is clear that this \( \delta \) will not be a function in the usual sense; mathematicians call the "function" we are about to define "distributions". Note that although they are infinite at \( x = x_0 \), the result of integrals over it will be finite always. So things are well-defined under \( \int \)'s.
In order to proceed we will define such a "function" as a limiting procedure. Consider the box car function

\[ B_a(x) = \begin{cases} \frac{1}{2a} & 1 \leq a \\ 0 & a \leq 1 \end{cases} \]

Which depends on parameter \( a \). As \( a \to 0 \) this will go to the desired function, note that it gets higher at \( x=a \) and it is zero almost everywhere (except at \( x=0 \)). Note that,

\[
\int_{-\infty}^{\infty} B_a(x) \, dx = \int_{-\infty}^{\infty} \frac{1}{2a} \, dx = \frac{1}{2a} \left[ x \right]_{-a}^{a} = \frac{1}{2a} (a-a) = 1
\]

So, as \( a \to 0 \) the area under \( B_a(x) \) is always unity. Furthermore,

\[
\int B_a(x-x_0) f(x) \, dx = \int B_a(y) f(y+x_0) \, dy = \frac{1}{2a} \int f(y+x_0) \, dy
\]

\[
= \frac{1}{2a} \int_{x_0-a}^{x_0+a} f(x) \, dx = \text{average of } f(x) \text{ about } x=x_0
\]

Note that indeed the left integral is only dependent on \( f(x) \) near \( x=x_0 \) if \( a \) is small, and it corresponds to the average of \( f(x) \) near \( x=x_0 \), by summing over the values in interval of width \( 2a \) (from \( x_0-a \) to \( x_0+a \)) and dividing by \( 2a \).

As \( a \to 0 \) the average of the function can only be \( f(x_0) \), i.e.

\[
\lim_{a \to 0} \int_{-\infty}^{\infty} B_a(x-x_0) f(x) \, dx = f(x_0) \equiv \int \delta_0(x-x_0) f(x) \, dx
\]

Such a limit defines what we call as the Dirac delta function \( \delta_0(x-x_0) \) centered at point \( x=x_0 \), i.e., \( \lim_{a \to 0} B_a(x-x_0) \to \delta_0(x-x_0) \).
We can see that indeed such definition leads to \( \varphi(x_0) \).

We need to show that

\[
\varphi = \lim_{a \to 0} \frac{1}{2a} \int_{x_0-a}^{x_0+a} f(x) \, dx = f(x_0)
\]

Let's expand in Taylor series about \( x = x_0 \), since the integral is dominated by contributions at \( x = x_0 \),

\[
f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{d^n f}{dx^n}(x_0) \right) (x-x_0)^n
\]

\[
= f(x_0) + f'(x_0) (x-x_0) + f''(x_0) \frac{(x-x_0)^2}{2} + \cdots
\]

\[
\Rightarrow \varphi = \lim_{a \to 0} \frac{1}{2a} \int_{x_0-a}^{x_0+a} \left[ f(x_0) + f'(x_0) (x-x_0) + \cdots \right] \, dx
\]

Now, the terms in Taylor series have dependence on \( a \) only through \( (x-x_0)^n \); so we need to compute the integrals

\[
\int_{x_0-a}^{x_0+a} (x-x_0)^n \, dx = \int_{-a}^{a} y^n \, dy = \left. \frac{y^{n+1}}{n+1} \right|_{-a}^{a} = \frac{a^{n+1} - (-a)^{n+1}}{n+1}
\]

\[
= \begin{cases} 
0 & n \text{ odd} \\
\frac{2}{n+1} a^{n+1} & n \text{ even}
\end{cases}
\]

Then we have,

\[
\varphi = \lim_{a \to 0} \frac{1}{2a} \left[ 2a f(x_0) + \frac{2a^3}{3} \frac{1}{2} f''(x_0) + \cdots \right]
\]

\[
= \lim_{a \to 0} \left[ f(x_0) + \frac{1}{6} a^2 f''(x_0) + \text{high powers of } a \right]
\]

\[
= f(x_0)
\]
The delta function can be defined as well as a limit of other functions (we shall see this later), not only $\delta_a(x)$ as $a \to 0$.

In all cases it involves of course a function that becomes infinite at $x = 0$ but whose area is constant (and equal to unity) as the parameter approaches its limit (e.g. $a \to 0$).

So, the main properties are,

\[
\begin{align*}
\int_{-\infty}^{\infty} \delta_a(x-x_0) \, dx &= 1 \\
\int_{-\infty}^{\infty} dx \, \delta_a(x-x_0) \, f(x) &= f(x_0)
\end{align*}
\]

In addition,

\[
\int_{-\infty}^{\infty} dx \, \delta_a \left[ c(x-x_0) \right] f(x) = \frac{f(x_0)}{|c|}
\]

we can see this easily. Assume $c > 0$, then,

\[
\int_{-\infty}^{\infty} dx \, \delta_a \left[ c(x-x_0) \right] f(x) = \int_{-\infty}^{\infty} \frac{dy}{c} \delta_a \left[ y-cx \right] f \left( \frac{y}{c} \right)
\]

\[
= \frac{1}{c} \int_{-\infty}^{\infty} dy \, \delta_a \left[ y-y_0 \right] f \left( \frac{y}{c} \right) = \frac{1}{c} \frac{f \left( \frac{y_0}{c} \right)}{|c|} = \frac{1}{c} f(x_0)
\]

For $c < 0$ we have similarly,

\[
\int_{-\infty}^{\infty} dx \, \delta_a \left[ c(x-x_0) \right] f(x) = -\int_{-\infty}^{\infty} \frac{dy}{c} \delta_a \left[ y-cx_0 \right] f \left( \frac{y}{c} \right) = -\frac{1}{c} \int_{-\infty}^{\infty} dy \, \delta_a \left[ y-y_0 \right] f \left( \frac{y}{c} \right)
\]

Note that since $c < 0$, when $x \to \infty$, $y \to -\infty$ and increases, thus the change in limit of integration. Then,
\[
\int_0^\infty dx \, \delta_0 \left[ c(x-x_0) \right] f(x) = \frac{1}{c} f(x_0)
\]

So indeed,
\[
\int_0^\infty dx \, \delta_0 \left[ c(x-x_0) \right] f(x) = \frac{f(x_0)}{1c}
\]

In other words,
\[
\delta_0 \left[ c(x-x_0) \right] = \frac{\delta_0 (x-x_0)}{1c}
\]

Similarly, we can calculate the delta function of a function, e.g.
\[
\int \delta_0 [g(x)] f(x) \, dx = ?
\]

Let \( g \) be zero for some \( x=x_0 \) \( g(x_0)=0 \) - obviously the \( \delta_0 [g(x)] \) will pick up the value of \( f(x) \) at \( x=x_0 \), that is
\[
\int \delta_0 [g(x)] f(x) \, dx = A \, f(x_0) \text{ when } g(x_0)=0
\]

for some proportionality constant \( A \), which we have to calculate.

Again, we can expand \( g(x) \) about \( x=x_0 \), since integral will be
dominated by contribution near \( x=x_0 \),
\[
g(x) = g(x_0) + g'(x_0) (x-x_0) + g''(x_0) \frac{(x-x_0)^2}{2} + \cdots
\]

\[= g'(x_0) (x-x_0) + \text{higher-order terms}\]

Then to first approximation,
\[
\int \delta_0 [g(x)] f(x) \, dx = \int \delta_0 \left[ g'(x_0) (x-x_0) \right] f(x) \, dx
\]
But now this is exactly as the previous case for \( c = g'(x_0) \), since \( g'(x_0) \) is a constant independent of \( x \), then

\[
\int_{-\infty}^{\infty} \delta_p \left[ g(x) \right] f(x) \, dx = \delta_p \left( \frac{f(x_0)}{|g'(x_0)|} \right)
\]

or:

\[
\delta_p \left[ g(x) \right] = \frac{\delta(x-x_0)}{|g'(x_0)|}
\]

where \( x_0 \) is such that \( g(x_0) = 0 \)

Now, the function \( g(x) \) may have lots of zeros, then each one contributes as well - If \( g(x_i) = 0 \) \( i = 1, \ldots, N \)

\[
\Rightarrow \quad \delta_p \left[ g(x) \right] = \sum_{i=1}^{N} \frac{\delta(x-x_i)}{|g'(x_i)|}
\]

\( g(x_i) = g(x_1) = \ldots = g(x_N) = 0 \)

So far we discussed things in 1D. In more than one dimension, things behave as you expect, e.g. in 3D we have

\[
\delta(x-F_0) = \delta(x-x_0) \delta(y-y_0) \delta(z-z_0)
\]

That is, a delta function in each dimension. So then

\[
\int \delta(x-F_0) f(x) \, d^3x = \iint \delta(x-x_0) \delta(y-y_0) \delta(z-z_0) f(x_0, y_0, z_0) \, dy \, dz
\]

\[
= \int \int \delta(y-y_0) \delta(z-z_0) f(x_0, y_0, z_0) \, dy \, dz = \int \int f(x_0, y_0, z) \delta(z-z_0) \, dz = f(x_0, y_0, z_0) = f(F_0)
\]
And we also have of course: \[ \int \delta_b(\mathbf{r} - \mathbf{r}_0) \, d^3r = 1 \]

Since this follows from previous work by letting \( f(\mathbf{r}) = 1 \).

In spherical coordinates, it is a bit more tricky. For example, on the surface of the sphere, we want that

\[ \int ds \, \delta_b(\mathbf{r} - \mathbf{r}_0) \, f(\mathbf{r}) = f(\mathbf{r}_0) \quad \text{where} \quad ds = \sin \theta \, d\theta \, d\phi \]

where \( \mathbf{r} \) is the unit vector that points in direction \((\theta, \phi)\):

\[ \mathbf{r} = \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix} \quad \text{and similarly for } \mathbf{r}_0 \]

\[ \int ds \, \delta_b(\mathbf{r} - \mathbf{r}_0) \, f(\mathbf{r}) = \int_0^\pi \sin \theta \, d\theta \int_0^{2\pi} d\phi \quad A \, \delta_b(\theta - \theta_0) \, \delta_b(\phi - \phi_0) \, f(\theta, \phi) \]

where we wrote \( \delta_b(\mathbf{r} - \mathbf{r}_0) = A \, \delta_b(\theta - \theta_0) \, \delta_b(\phi - \phi_0) \); i.e., a delta function in each dimension times a constant we have to figure out.

We see that to recover \( f(\theta_0, \phi_0) \), we need \( A \sin \theta = 1 \), for then

\[ \int ds \, \delta_b(\mathbf{r} - \mathbf{r}_0) \, f(\mathbf{r}) = \int_0^\pi d\theta \int_0^{2\pi} d\phi \quad \delta_b(\theta - \theta_0) \quad \delta_b(\phi - \phi_0) \quad f(\theta, \phi) = f(\theta_0, \phi_0) = f(r_0) \]

Then we have in the surface of the sphere,

\[ \delta_b(\mathbf{r} - \mathbf{r}_0) = \frac{\delta_b(\theta - \theta_0) \, \delta_b(\phi - \phi_0)}{\sin \theta} \quad \text{(spherical 2D)} \]

We can similarly work out the 3D case in spherical coordinates:
We want
\[ \int \delta_D (\mathbf{r} - \mathbf{r}_0) \ f(\mathbf{r}) \ d^3r = f(r_0) \]

But in spherical coordinates, \( d^3r = r^2 \sin \theta \ d\theta \ d\phi \ d\rho \)

Let's write again \( \delta_D (\mathbf{r} - \mathbf{r}_0) = A \ \delta_D (r - r_0) \ \delta_D (\theta - \theta_0) \ \delta_D (\psi - \psi_0) \)

Then:
\[ \int A r^2 \sin \theta \ \delta_D (r - r_0) \ \delta_D (\theta - \theta_0) \ \delta_D (\psi - \psi_0) \ f(r_0, \theta, \psi) \ dr \ d\theta \ d\phi = f(r_0, \theta_0, \psi_0) \]

\[ \Rightarrow A r^2 \sin \theta = 1 \]

So that:
\[ \delta_D (\mathbf{r} - \mathbf{r}_0) = \frac{\delta_D (r - r_0) \ \delta_D (\theta - \theta_0) \ \delta_D (\psi - \psi_0)}{r^2 \sin \theta} \]  

(Spherical 3D)