Gauss' Theorem

Remember that last class we derived the flux \( \Phi \) of a vector field \( \vec{V} \)

\[
\Phi = \oint_S \vec{V} \cdot d\vec{s}
\]

over a closed surface \( S \) (also written as \( \iint_S \vec{V} \cdot d\vec{s} \) sometimes), and found that for an infinitesimal cube of volume \( dV \), this was proportional to the divergence of \( \vec{V} \), \( \nabla \cdot \vec{V} \),

\[
\Phi = \nabla \cdot \vec{V} \left. \right|_{S} dV
\]

This is in general true, for any surface \( S \) that encloses volume \( V \), we have Gauss' Theorem (also known as Divergence Theorem),

\[
\iiint_V \nabla \cdot \vec{V} \, d^3V = \iiint_V \nabla \cdot \vec{V} \, d^3V \\
\text{or: } \oint_S \vec{V} \cdot d\vec{s} = \iiint_V \nabla \cdot \vec{V} \, d^3V \\
\text{With the volume } V(S) \text{ is that enclosed by the closed surface } S, \text{ with normal vector } \hat{n} \text{ so that } d\vec{s} = dS \hat{n} \text{ at each point, } \hat{n} \text{ points out of the closed surface } S. \\

Example: Consider \( \vec{V} = (x, y, z) \) and \( S \) the surface of a cylinder of radius \( r \) and height \( h \) - let's check the Gauss' Theorem:

\[
\nabla \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 1 + 1 + 1 = 3
\]

\[
\Rightarrow \int_{V} 3 \, d^3V = 3V = 3 \times \pi r^2 h
\]

Now consider the integral for the flux of \( \vec{V} \) over the cylinder surface...
\[
\int \vec{V} \cdot d\mathbf{s} = \int \vec{V} \cdot d\mathbf{s}_{\text{bottom}} + \int \vec{V} \cdot d\mathbf{s}_{\text{top}} + \int \vec{V} \cdot d\mathbf{s}_{\text{side}}
\]

The first contribution is zero, because \( \vec{V} \) at the surface is zero, since
\[
\vec{V} = (x, y, 0) \quad d\mathbf{s}_{\text{bottom}} = ds_{(0,0,-a)} \Rightarrow \vec{V} \cdot d\mathbf{s} = 0
\]

The second is non-zero
\[
\vec{V}_{\text{top}} = (x, y, h) \quad d\mathbf{s}_{\text{top}} = ds_{(0,0,1)} \Rightarrow \vec{V} \cdot d\mathbf{s} = h \, ds
\]
\[
\Rightarrow \int \vec{V} \cdot d\mathbf{s}_{\text{top}} = h \int ds_{\text{top}} = h \pi a^2
\]

For the contribution from the walls of the cylinder we have:
\[
\vec{V}_{\text{side}} = (x, y, z) \quad \text{where} \quad x^2 + y^2 = a^2
\]
\[
d\mathbf{s}_{\text{side}} = ds \quad \hat{n} = \frac{(x, y, 0)}{\sqrt{x^2 + y^2}} = \frac{1}{a} (x, y, 0) \quad \text{since} \quad \hat{n} \text{ is on the } xy \text{ plane (and normalized to unity)}
\]
\[
\Rightarrow \vec{V} \cdot d\mathbf{s} = ds \frac{x^2 + y^2}{a} = a \, ds
\]
\[
\Rightarrow \int \vec{V} \cdot d\mathbf{s}_{\text{side}} = a \int ds_{\text{side}} = 2\pi ah
\]

Summing up,
\[h \pi a^2 + a^2 \pi h = 3 \pi a^2 h, \text{ which agrees with } \int \vec{V} \cdot d\mathbf{V} = \int_{V(S)}
\]

Example 2: Using Gauss' theorem to calculate gravity and electric fields.

As we discussed last time for a fluid, the fact that the flux is non-zero across a closed surface \( S \) means that either

i) there is a source or a sink inside \( V(S) \)

ii) the density is changing with time, i.e. not the same amount of fluid that goes in goes out.
Let's consider the case when things are time independent (we will come back to this when we study applications to conservation laws). Then, if the flux is changing, it means that there is either a source or a sink, that means that

$$\nabla \cdot \mathbf{F} = \Psi(f)$$

Where \(\Psi(f)\) is some scalar function. If \(\Psi > 0\), then it is a source, and fluid lines go outward from the source; otherwise, if \(\Psi < 0\) lines converge at the sink.

![Source and Sink Diagrams]

Clearly, for a source the flux \(\Phi > 0\), since \(\nabla \cdot \mathbf{F} > 0\)
for a sink \(\Phi < 0\)

In gravity and electromagnetism, we can describe things in a very similar way. For example,

$$\begin{cases} \nabla \cdot \mathbf{g} = -4\pi G \rho(r) \quad \mathbf{g} : \text{gravity field} \quad G: \text{Newton's constant} \\ \nabla \cdot \mathbf{E} = 4\pi \rho(r) \quad \mathbf{E} : \text{electric field} \quad \rho(r) : \text{charge density} \end{cases}$$

As expected, mass density is a sink of gravitational field, charge density is a source of electric field.

[Note: The difference in sign in the equations, can you explain why this is so?]

We can calculate the gravitational (electric) field by using Gauss:

$$\oint_{S} \mathbf{g} \cdot d\mathbf{A} = \int_{V} \nabla \cdot \mathbf{g} \, d^{3}V$$

$$\oint_{S} \mathbf{E} \cdot d\mathbf{A} = \int_{V} \nabla \cdot \mathbf{E} \, d^{3}V$$
Imagine we have a spherically symmetric mass (or charge) distribution inside some surface (which we take to be spherical), then

\[ ds^2 = ds^2, \quad \vec{g} = g_{\mu\nu} \hat{n} \quad \text{by symmetry} \]

\[ \int \vec{g} \cdot ds = 4\pi R^2 g(r) = \int \frac{-4\pi G M(r)}{r} \, d\tau \]

\[ \Rightarrow \quad g(r) = -\frac{G}{R^2} \int f(r) \, d\tau = -\frac{GM(r)}{R^2} \quad \text{while } M(r) \text{ is the enclosed mass inside sphere of radius } R. \]

So, at any distance \( R \) from the mass distribution the gravitational force depends on the enclosed mass divided by distance squared - [Similarly for the electric field]. Note that the gravitational field depends only on the enclosed mass, so two distributions with same enclosed mass generate the same field outside, e.g.

The ball of mass \( M \) (1) and the hollow sphere of total mass \( M \) (2)

generate the same field \( \vec{g}_1(r) = \vec{g}_2(r) = -\frac{GM}{r^2} \hat{r} \) at large distances. As expected, the force on a mass \( m \) at \( r \) is

\[ \vec{F} = -\frac{GMm}{r^2} \hat{r} \]

\( (F = mg) \quad \text{(same as for electric field)} \]

\[ \vec{E} = \vec{q} \cdot \vec{E} = \vec{q} \cdot \text{charge} \]

Note that the surface we use to calculate the gravity field respects the symmetry of the distribution, to simplify the calculation (i.e., to have \( \vec{g} \) and \( ds^2 \) point parallel to each other). In problems where the symmetry is not spherical, one must use a different surface to make calculation easier.
Stokes Theorem

Last class we discussed the circulation of a vector field \( \mathbf{F} \),

\[
\text{circulation} = \oint_{C} \mathbf{F} \cdot d\mathbf{r}
\]

along a closed curve \( C \), and found that it was given by the perpendicular component of curl of the vector field \( \mathbf{F} \) times the area, \( \text{perpendicular component} \)

\[
(\nabla \times \mathbf{F}) \cdot \mathbf{n} \, ds = \oint_{C} \mathbf{F} \cdot d\mathbf{r}
\]

We can write this in integral form by integrating over \( ds \),

\[
\oint_{C} \mathbf{F} \cdot d\mathbf{r} = \iint_{S(C)} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, ds
\]

Stokes' Theorem

Which says that the circulation of \( \mathbf{F} \) along a curve \( C \), is given by the flux of its curl \( \nabla \times \mathbf{F} \) across the (open) surface \( S(C) \) which is bounded by the curve \( C \). Note that: the normal to surface \( \mathbf{n} \) is defined from the direction of the circulation using the right-hand rule, eg.

It is also easy to show that the surface can be chosen arbitrarily, since they are many surfaces that are bounded by the same curve \( C \) - let's see
\[
\oint_s (\mathbf{a} \cdot d\mathbf{s}) = \int_{s_1} (\mathbf{a} \cdot d\mathbf{s}_1) - \int_{s_2} (\mathbf{a} \cdot d\mathbf{s}_2)
\]

This is true, if and only if

\[
\int_{s_1} (\mathbf{a} \cdot d\mathbf{s}_1) - \int_{s_2} (\mathbf{a} \cdot d\mathbf{s}_2) = 0
\]

We can change the sign of \(d\mathbf{s}_2\) to make it outward, thus:

\[
\int_{s_1} (\mathbf{a} \cdot d\mathbf{s}_1) + \int_{s_2} (\mathbf{a} \cdot d\mathbf{s}_2) = 0 \quad \text{where} \quad \begin{cases}
\quad \int_{s_1} \mathbf{a} \cdot d\mathbf{s}_1 = d\mathbf{s}_1 \\
\quad \int_{s_2} \mathbf{a} \cdot d\mathbf{s}_2 = -d\mathbf{s}_2
\end{cases}
\]

\[
\oint_s (\mathbf{a} \cdot d\mathbf{s}) = \int_{s_1} (\mathbf{a} \cdot d\mathbf{s}_1) + \int_{s_2} (\mathbf{a} \cdot d\mathbf{s}_2) = 0
\]

This is what we have to prove. But by Gauss Theorem,

\[
\oint_s (\mathbf{a} \cdot d\mathbf{s}) = \text{flux of vector } \nabla \times \mathbf{a} \quad \text{across closed surface } S
\]

Now:

\[
\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \begin{vmatrix}
\mathbf{e}_i & \mathbf{e}_j & \mathbf{e}_k \\
\partial_i & \partial_j & \partial_k \\
\mathbf{a}_i & \mathbf{a}_j & \mathbf{a}_k
\end{vmatrix} = \begin{vmatrix}
e_{ijk} & \partial_j & \partial_k \\
\mathbf{a}_i & \mathbf{a}_j & \mathbf{a}_k
\end{vmatrix}
\]

Then, we showed that the surface chosen doesn't change results as long as it is bounded by \(C\).

Example:

\[
\mathbf{a} = (4y, x, 2z)
\]

Calculate \(\oint_s (\mathbf{a} \cdot d\mathbf{s})\) for \(S_1\), a hemisphere: \(x^2 + y^2 + z^2 = a^2, z \geq 0\)

By Stokes' theorem, the integral over open surface \(S_1\) is same as circulation of \(\mathbf{a}\) along curve \(C_1\), which is a circle: \(x^2 + y^2 = a^2\) –
but also, again by Stokes theorem, integral is also flux across open surface $S_2$, a circle $x^2 + y^2 = a^2$, which is bounded by $S$.

Let's calculate all and see they are the same.

$$
\oint \mathbf{v} \cdot d\mathbf{S} = \int_0^{2\pi} \mathbf{a} \hat{\theta} \cdot \mathbf{v}^2
$$

$v^2 = (x, y, 0)$ @ arcl, $\sin \theta = 0$

$\hat{\theta} \cdot v^2 = \frac{a}{2} \cos \theta - 4y \sin \theta$

but $\begin{cases} x = a \cos \theta \\ y = b \sin \theta \end{cases}$

$\Rightarrow \hat{\theta} \cdot v^2 = a \left( \cos ^2 \theta - 4 \sin \theta \right) = a \left( 1 - 5 \sin ^2 \theta \right)$

$$
\oint \mathbf{v} \cdot d\mathbf{S} = \int_0^{2\pi} a^2 \left( 1 - 5 \sin ^2 \theta \right) = a^2 \times \left[ 2\pi - 5 \int_0^{\pi/2} \frac{\sin 2\theta}{\sin \theta} d\theta \right]
$$

average of $\sin \theta$ over a period $\frac{\pi}{2}$

$$
= 2\pi a^2 \left( 1 - \frac{5}{2} \right) = 2\pi a^2 \left( -\frac{3}{2} \right) = -3\pi a^2
$$

$$
\int \left( \nabla \times \mathbf{v} \right) \cdot dS = ?
$$

$\nabla \times v^2 = -3 \hat{z}$

$dS = r^2 \sin \phi \, d\phi \, d\phi$

$r = \left( x^2 + y^2 \right)^{1/2}$

$$
(\nabla \times \mathbf{v}) \cdot dS = -3 \frac{a^2}{2} \int_0^{\pi/2} \int_0^{2\pi} \sin \phi \, \cos \phi \, d\phi \, d\phi
$$

$$
= -3 a^2 \int_0^{\pi/2} \sin \phi \, \cos \phi \, d\phi = -3 a^2 \int_0^{\pi/2} \sin \phi \, \cos \phi \, d\phi
$$

$$
= -3 a^2 \left[ \frac{1}{2} (\cos^2 \theta) \right]_0^{\pi/2} = -3 a^2 \pi
$$

$$
= -3 a^2 \pi
$$

$$
\int_S (\nabla \times \mathbf{v}) \cdot dS = -3 a^2 \pi
$$
Finally, the easiest is to use $S_2$:

\[
\begin{aligned}
\int_{S_2} \left( \nabla \times \mathbf{v} \right) \cdot \mathbf{n} dS &= \int_0^a r \, dr \int_0^{2\pi} (-3 \hat{z}) \cdot \hat{z} \, d\theta \\
&= -3 \int_0^a \frac{a^2}{2} \, d\theta = -3a^2 \pi
\end{aligned}
\]