Potential Theory: Helmholtz's Theorem

We are now going to take advantage of two simple facts (which you should make sure you understand): 1) $\nabla \times \nabla \phi = 0$ and 2) $\nabla \cdot (\nabla \times A) = 0$

Conservative Fields: Potential Theory

The definitions here derive from particle mechanics where a force is said to be conservative if the work done by displacing a particle from point A to point B is independent of the path that goes from A to B:

$$\text{Work} = \int_{A \to B} F \cdot d\ell$$

Recall that when we discussed gradients we showed that the gradient of a function $\nabla \phi$ has this property of path independence--let's explore this more.

If the integral by path 1 and 2 give the same answer, we can construct a curve $C$ that goes on path 1 from A to B and comes back from B to A on "path 2" (which means to travel path 2 backwards), so:

$$\oint_{C} F \cdot d\ell = 0 = \int_{A \to B} F \cdot d\ell + \int_{B \to A} F \cdot d\ell$$

If work is independent of path, and since it changes sign when A and B are interchanged, we have:
Work \[= \int_{A}^{B} \mathbf{F} \cdot d\mathbf{r} = \Phi(A) - \Phi(B) \]

if we choose \( A \) and \( B \) close to each other, we have the infinitesimal relation

\[ \mathbf{F} \cdot d\mathbf{r} = - (\Phi(B) - \Phi(A)) = -d\Phi = -\nabla\Phi \cdot d\mathbf{r} \]

"final" \hspace{1cm} "initial"

Thus we obtain \[ \mathbf{F} = -\nabla\Phi \]

as a result of path independence (or conservative force \( \mathbf{F} \)).

Obviously, the converse is also true, i.e.

if \( \mathbf{F} = -\nabla\Phi \Rightarrow \oint_{C} \mathbf{F} \cdot d\mathbf{r} = 0 \Rightarrow \) path independence for Work \( A \rightarrow B \)

Also, since \( \nabla \times \nabla \Phi = 0 \) then \( \nabla \times \mathbf{F} = 0 \) if \( \mathbf{F} = -\nabla\Phi \) - And from Stokes theorem:

\[ \oint_{C} \mathbf{F} \cdot d\mathbf{r} = 0 = \iint_{S(C)} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \]

So if \( \oint_{C} \mathbf{F} \cdot d\mathbf{r} = 0 \) \( \Rightarrow \nabla \times \mathbf{F} = 0 \) (since \( C \) and thus \( S(C) \) is) arbitrary

and also \( \nabla \times \mathbf{F} = 0 \Rightarrow \oint_{C} \mathbf{F} \cdot d\mathbf{r} = 0 \)

[Note: we are assuming that regions of space where these integrals are done are simply connected]

Thus we have:

Thus we have:

\[ \oint_{C} \mathbf{F} \cdot d\mathbf{r} = 0 \]

\[ \mathbf{F} = -\nabla\Phi \]

\[ \nabla \times \mathbf{F} = 0 \] "curl free"

\[ \Phi: \text{Scalar potential} \]

\[ \nabla \times (\nabla \Phi) = 0 \]

"path independence"
Therefore, any conservative field can be written as the gradient of some scalar potential $\phi$. The divergence of such force will then be

$$\nabla \cdot F = -\nabla \cdot (\nabla \phi) = -\nabla^2 \phi$$

where $\nabla^2$ is the Laplacian we discussed, i.e. in cartesian co-ords.

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

in 3D.

From this we have that $\nabla^2 \phi$ acts as a source ($\nabla^2 \phi > 0$) or sink ($\nabla^2 \phi < 0$) for $F$.

**Solenoidal Fields**

What about $\nabla \times F$? When $F = -\nabla \phi$, $\nabla \times F = 0$, so there is no "vorticity" in $F$, nor circulation — the flow described by $F$ (plot the field lines) is just a pure divergence, so sink and source describe the field $F$ configuration.

We now explore the opposite case, when we have a flow/field with zero divergence (no sources nor sinks) but non-zero curl. Such vector fields are called solenoidal (or rotational).

One example of such fields is the magnetic field (in the static case) which you are familiar with. So let's denote the vector field by $\vec{B}$, we have

$$\nabla \cdot \vec{B} = 0 \quad \text{but} \quad \nabla \times \vec{B} \neq 0$$
Now we can use the identity $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ to write down
\[ \mathbf{B} = \nabla \times \mathbf{A} \iff \nabla \cdot \mathbf{B} = 0 \]
which automatically satisfies $\nabla \cdot \mathbf{B} = 0$. $\mathbf{A}$ is known as the vector potential.

Note that if we are given $\mathbf{B}$, we can find an infinite number of $\mathbf{A}$'s such that $\mathbf{B} = \nabla \times \mathbf{A}$. Indeed, if $\mathbf{A}_1$ is such that $\nabla \times \mathbf{A}_1 = \mathbf{B}$ then
\[ \mathbf{A}_2 = \mathbf{A}_1 + \nabla \phi \]
where $\phi$ is arbitrary scalar function of coords.

Also satisfies $\nabla \times \mathbf{A}_2 = \nabla \times \mathbf{A}_1 + \nabla \times \nabla \phi = \mathbf{B}$

Thus we say that the vector potential $\mathbf{A}$ is "determined up to a gradient".

Finally, we discussed potential (zero-curl) vector fields and solenoidal (zero divergence) vector fields, but a special case of these are fields for which both the curl and divergence vanish, such are sometimes called Laplacian fields. So a Laplacian vector field $\mathbf{C}$ satisfies $\nabla \times \mathbf{C} = 0 \Rightarrow \mathbf{C} = -\nabla \phi$
and also $\nabla \cdot \mathbf{C} = 0 \Rightarrow \nabla^2 \phi = 0$ (Laplace's equation)

i.e. the corresponding scalar potential obeys Laplace's equation.
[You can convince yourself that had we started from $\mathbf{C} = \nabla \times \mathbf{A}$ we would have reached the same conclusion]
Note: Even though $\nabla \cdot \vec{\omega} = 0$ & $\nabla \times \vec{\omega} = 0$, it does not mean that $\vec{\omega}$ vanishes; the solutions of Laplace's equation are typically non-zero and non-trivial and depend only on the boundary conditions. We will in fact explore such solutions later in the course in much detail.

**HELMHOLTZ DECOMPOSITION THEOREM**

This important theorem says that the two cases we considered above (potential and solenoidal vector fields) are a "basis," in the sense that any vector field can be written in terms of a potential part and a solenoidal part.

Before showing the decomposition theorem, let's prove the following statement about uniqueness:

- A vector is uniquely specified by giving its divergence and its curl within a region and its normal component over the boundary.

Proof: Let's assume that

\[ \int \nabla \cdot \vec{\nu}_1 = 0 \]

\[ \int \nabla \times \vec{\nu}_1 = \vec{\omega} \]

and that $\vec{\nu}_1 \cdot \vec{n} = \nu_{1n}$ at the boundary. If there were a second vector $\vec{\nu}_2$ satisfying these same conditions, we want to see that $\vec{\nu}_2 = \vec{\nu}_1$, therefore proving that it is unique. So let

\[ \vec{d} = \vec{\nu}_1 - \vec{\nu}_2 \]

Then we have by definition:
\[ \begin{cases} \nabla \cdot \vec{a} = 0 \\ \nabla \times \vec{a} = 0 \\ \vec{a} \cdot \vec{n} = d_n = 0 & \text{boundary} \end{cases} \]

Since \( \vec{a} \) is curl-free we may write

\[ \vec{J} = -\nabla \phi \Rightarrow \nabla \cdot \vec{a} = -\nabla \phi = 0 \quad \text{(Laplace's eqn.)} \]

To show that \( \vec{J} \) is the zero vector (and thus \( \vec{U_1} = \vec{U_2} \)) we will use Green's theorem:

\[ \oint_S \phi_1 \nabla \phi_2 \cdot d\vec{s} = \int_V (\phi_1 \nabla^2 \phi_2 + \phi_2 \nabla \phi_1) \, d^3r \]

which can be easily shown from Gauss' theorem. Since,

\[ \nabla \cdot (\phi_1 \nabla \phi_2) = \nabla \phi_1 \cdot \nabla \phi_2 + \phi_1 \nabla^2 \phi_2 \]

we have from Gauss that

\[ \oint_V (\phi_1 \nabla^2 \phi_2 + \nabla \phi_1 \cdot \nabla \phi_2) \, d^3r = \int_V \nabla \cdot (\phi_1 \nabla \phi_2) \, d^3r = \oint_S \phi_1 \nabla \phi_2 \cdot d\vec{s} \]

Now, let's take \( \phi_1 = \phi_2 = \phi \) with \( \vec{a} = -\nabla \phi \). We have (since \( \nabla^2 \phi = 0 \))

\[ \oint_V \vec{a} \cdot d\vec{r} = \oint V \nabla \phi \cdot d\vec{r} = \oint S \phi \nabla \phi \cdot dS = -\int \phi (\vec{a} \cdot \vec{n}) \, dS = 0 \quad \text{\(d_n = 0 \) @ boundary} \]

Since \( \vec{a} \cdot \vec{a} = \vec{a}^2 \geq 0 \), the only way for this to hold is that

\[ \vec{a} = 0 \Rightarrow \vec{U}_1 = \vec{U}_2 \Rightarrow \vec{U}_1 \text{ is unique.} \]

Now we are ready to show Helmholtz theorem: A vector \( \vec{b} \) with divergence \( \phi \) and vanishing curl \( \vec{a} \) vanishing at infinity can be written as the sum of two vector fields: one
potential and one solenoidal, i.e.

\[
\begin{cases}
\nabla \cdot \vec{v} = \Theta (F) \\
\nabla \times \vec{v} = \vec{\omega} (F)
\end{cases}
\Rightarrow \quad \vec{v} = - \nabla \phi + \nabla \times \vec{A}
\]

and Helmholtz tells us that \( \phi \) and \( \vec{A} \) are given by:

\[
\begin{cases}
\phi (F) = \frac{1}{4\pi} \int \frac{\Theta (F')}{|F - F'|} d^3r' \\
\vec{A} (F) = \frac{1}{4\pi} \int \frac{\vec{\omega} (F')}{|F - F'|} d^3r'
\end{cases}
\]

From the uniqueness theorem discussed above, \( \vec{v} \) is uniquely determined by \( \Theta \) and \( \vec{\omega} \) and the boundary conditions (\( \Theta \) and \( \vec{\omega} \) vanishing at infinity). So all we have to show is

\[
\begin{cases}
\nabla \cdot \vec{v} = \Theta = - \nabla^2 \phi \quad (\text{since } \nabla \cdot (\nabla \times \vec{A}) = 0) \\
\nabla \times \vec{v} = \vec{\omega} = \nabla \times (\nabla \times \vec{A}) \quad (\text{since } \nabla \times \nabla \phi = 0)
\end{cases}
\]

So, \( \Theta \) determines \( \phi \) and \( \vec{\omega} \) determines \( \vec{A} \).

Let's verify the scalar potential equation first. Want to check that:

\[
- \nabla^2 \phi (F) = - \nabla^2 \frac{1}{4\pi} \int \frac{\Theta (F')}{|F - F'|} d^3r' = \Theta (F)
\]

Note that the Laplacian acts on the \( F \) variables (not \( F' \)). So we need to calculate \( \nabla^2 \frac{1}{|F - F'|} \).
We show below that

\[ \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -4\pi \delta_D(\mathbf{r} - \mathbf{r}') \]  

(\star)

where \( \delta_D(\mathbf{r}) \) is the "Dirac delta function." We will discuss this object in detail later, for now we note the main properties:

\[
\begin{aligned}
\delta_D(\mathbf{r}) &= 0 \quad \text{if } \mathbf{r} \neq 0 \\
\int_V \delta_D(\mathbf{r}) \, d^3r &= 1 \quad \text{if volume } V \text{ includes the origin} \\
\int_V \delta_D(\mathbf{r} - \mathbf{r}_0) f(\mathbf{r}) \, d^3r &= f(\mathbf{r}_0) \quad \text{for arbitrary } f(\mathbf{r})
\end{aligned}
\]

So using these we have:\n
\( -\nabla^2 \phi = -\nabla^2 \frac{1}{4\pi} \int \Theta(\mathbf{r} - \mathbf{r}') \, d^3r' = \int \Theta(\mathbf{r} - \mathbf{r}') \delta_D(\mathbf{r} - \mathbf{r}') \, d^3r' = \Theta(\mathbf{r}) \)

so the scalar potential piece works out!

For the vector potential piece it works out along the same lines. First, we have:\n
\[ \nabla \times \mathbf{A} = \nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mathbf{\omega} \]

homework want to check

Now, from the definition of \( \mathbf{A} \) in terms of \( \mathbf{\omega} \), you can show that \( \nabla (\nabla \cdot \mathbf{A}) = 0 \) by integrating by parts, using Gauss theorem and that \( \nabla \cdot \mathbf{\omega} = 0 \) and that \( \mathbf{\omega} \) vanishes at infinity \( \Rightarrow \) we just need to show that
\[ -\nabla^2 \bar{A} = \bar{\omega} \]

but this again works out due to \( \delta_0 \) properties,

\[ -\nabla^2 \bar{A} = -\frac{1}{4\pi} \nabla^2 \int \frac{\bar{\omega}(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3r = \int \bar{\omega}(\vec{r}') \delta_0(|\vec{r}-\vec{r}'|) d^3r = \bar{\omega}(\vec{r}) \]

so the theorem is proved. Now, we still have to show (x). Let's consider the case \( F' = 0 \), since \( F' \neq 0 \) can be trivially restored. So we want to see that

\[ \nabla^2 \frac{1}{r} = -4\pi \delta_D(F) \]

which should 1) vanish if \( r \to 0 \) (according to \( \delta_0 \) properties)

2) should integrate to \(-4\pi\) if the volume of integration includes the origin (again due to \( \delta_0 \) properties)

1) \[ \nabla^2 \left( \frac{1}{r} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \frac{\partial}{\partial r} \left( \frac{1}{r} \right) \right] = \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \left( -\frac{1}{r^2} \right) \right] \]

\[ \frac{\partial}{\partial r} \left[ -1 \right] = 0 \]

\[ r \to 0 \]

2) \[ \int \nabla^2 \left( \frac{1}{r} \right) d^3r = \int \nabla \cdot \nabla \left( \frac{1}{r} \right) d^3r = \int \frac{\vec{\nabla} \cdot \vec{A}}{r} d\vec{s} \]

volume about the origin \( r = 0 \)

Now, since \( \nabla \left( \frac{1}{r} \right) \nabla \cdot -\frac{1}{r^2} \)

spherical coods
and let's assume, $V$ is a sphere very very close to $r = 0$ with radius $R \rightarrow 0$

$$\int_V \nabla^2 \left( \frac{1}{r} \right) d^3r = -\int_{\partial V} \frac{\mathbf{F}}{R^2} \cdot d\mathbf{S} = -\int_{\partial V} \frac{\mathbf{F} \cdot \mathbf{n}}{R^2} dS = -\frac{1}{R^2} \int_{\partial V} dS = -4\pi \sqrt{R}$$

Examples

1) The very important use of Helmholtz is electromagnetism, i.e. Maxwell equations:

$$\begin{cases} \nabla \cdot \mathbf{E} = 4\pi j \\ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \frac{4\pi}{c} j + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \end{cases}$$

for electric field $\mathbf{E}$, magnetic field $\mathbf{B}$, charge density $j$, and current density $j$. Note that

i) we specify divergence and curls of $\mathbf{E}, \mathbf{B}$ to solve for them

ii) in the static case, $\frac{\partial \mathbf{B}}{\partial t} = 0$, electric fields are potential

iii) magnetic fields are always solenoidal

2) A second example lets consider a simple case of Helmholtz decomposition. Let $\mathbf{F} = (x-y) \hat{e}_1 + (xy) \hat{e}_2$
for $\vec{e}_1 = \hat{x}$ and $\vec{e}_2 = \hat{y}$.

We have $\nabla \cdot \vec{F} = 2$ and $\vec{\nabla} \times \vec{F} = 2 \hat{z}$, so this field has both potential and solenoidal components. Its field lines look like

which by Helmholtz can be decomposed into

$$F_1 = -y \hat{x} + x \hat{y}$$
$$\nabla \times F_1 = 2 \hat{z}$$
$$\nabla \cdot F_1 = 0$$

$$\vec{F}_2 = x \hat{x} + y \hat{y}$$
$$\nabla \times \vec{F}_2 = 0$$
$$\nabla \cdot \vec{F}_2 = 2$$

solenoidal
(curling) field
(no sources nor sinks)

potential field
(conservative)
(source @ origin)