The surface-integral which we have to find is
\[ \int \int \left( l \xi + m \eta + n \zeta \right) \, dS; \]

(4)
or, substituting the values of \( \xi, \eta, \zeta \) in terms of \( X, Y, Z, \)
\[ \int \int \left( n \frac{dX}{ds} - m \frac{dY}{ds} + l \frac{dZ}{ds} \right) \, dS. \]

The part of this which depends on \( X \) may be written
\[ \int \int \left( \frac{dX}{ds} \frac{dx}{d\beta} + \frac{dX}{d\alpha} \frac{dy}{d\alpha} \right) d\beta \, d\alpha; \]
adding and subtracting \( \frac{dX}{ds} \frac{dx}{d\beta} \), this becomes
\[ \int \int \left( \frac{dX}{ds} \frac{dx}{d\beta} + \frac{dX}{d\alpha} \frac{dy}{d\alpha} \right) d\beta \, d\alpha; \]

(7)
and
\[ \int \int \left( \frac{dX}{ds} \frac{dx}{d\beta} - \frac{dX}{d\alpha} \frac{dy}{d\beta} \right) d\beta \, d\alpha. \]

(8)

Let us now suppose that the curves for which \( \alpha \) is constant form a series of closed curves surrounding a point on the surface for which \( \alpha \) has its minimum value, \( \alpha_0 \), and let the last curve of the series, for which \( \alpha = \alpha_1 \), coincide with the closed curve \( s \).

Let us also suppose that the curves for which \( \beta \) is constant form a series of lines drawn from the point at which \( \alpha = \alpha_0 \) to the closed curve \( s \), the first, \( \beta_0 \), and the last, \( \beta_1 \), being identical.

Integrating (8) by parts, the first term with respect to \( \alpha \) and the second with respect to \( \beta \), the double integrals destroy each other and the expression becomes
\[ \int_{\beta_0}^{\beta_1} \left( X \frac{dx}{d\beta} \right) d\beta - \int_{\alpha_0}^{\alpha_1} \left( X \frac{dx}{d\alpha} \right) d\alpha. \]

(9)

Since the point \( (\alpha, \beta) \) is identical with the point \( (\alpha, \beta_0) \), the third and fourth terms destroy each other; and since there is

but one value of \( x \) at the point where \( \alpha = \alpha_0 \), the second term is zero, and the expression is reduced to the first term:

The part of this which depends on \( X \) may be written
\[ \int \int \left( n \frac{dX}{ds} - m \frac{dY}{ds} + l \frac{dZ}{ds} \right) dS; \]
adding and subtracting \( \frac{dX}{ds} \frac{dx}{d\beta} \), this becomes
\[ \int \int \left( \frac{dX}{ds} \frac{dx}{d\beta} + \frac{dX}{d\alpha} \frac{dy}{d\alpha} \right) d\beta \, d\alpha; \]

(7)
and
\[ \int \int \left( \frac{dX}{ds} \frac{dx}{d\beta} - \frac{dX}{d\alpha} \frac{dy}{d\beta} \right) d\beta \, d\alpha. \]

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Let us also suppose that the curves for which \( \beta \) is constant form a series of lines drawn from the point at which \( \alpha = \alpha_0 \) to the closed curve \( s \), the first, \( \beta_0 \), and the last, \( \beta_1 \), being identical.

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(9)

Since the point \( (\alpha, \beta) \) is identical with the point \( (\alpha, \beta_0) \), the third and fourth terms destroy each other; and since there is

but one value of \( x \) at the point where \( \alpha = \alpha_0 \), the second term is zero, and the expression is reduced to the first term:

Since the curve \( x = x_0 \) is identical with the closed curve \( s \), we may write the expression in the form
\[ \int \frac{dX}{ds} \, ds, \]

(10)
where the integration is to be performed round the curve \( s \). We may treat in the same way the parts of the surface-integral which depend upon \( Y \) and \( Z \), so that we get finally,
\[ \int \left( (\xi + m \eta + n \zeta) \right) dS = \int \left( X \frac{dx}{ds} + Y \frac{dy}{ds} + Z \frac{dz}{ds} \right) ds; \]

(11)
where the first integral is extended over the surface \( S \), and the second round the bounding curve \( s \).

On the effect of the operator \( \nabla \) on a vector function.

25. We have seen that the operation denoted by \( \nabla \) is that by which a vector quantity is deduced from its potential. The same operation, however, when applied to a vector function, produces results which enter into the two theorems we have just proved (III and IV). The extension of this operator to vector displacements, and most of its further development, are due to Professor Tait.

Let \( \sigma \) be a vector function of \( \rho \), the vector of a variable point. Let us suppose, as usual, that
\[ \rho = ix + jy + kz, \]
and
\[ \sigma = iX + jY + kZ; \]
where \( X, Y, Z \) are the components of \( \sigma \) in the directions of the axes.

We have to perform on \( \sigma \) the operation
\[ \nabla = i \frac{d}{dx} + j \frac{d}{dy} + k \frac{d}{dz}. \]
Performing this operation, and remembering the rules for the multiplication of \( i, j, k \), we find that \( \nabla \sigma \) consists of two parts, one scalar and the other vector.

* This theorem was given by Professor Stokes, Smith's Prize Examination, 1854, question 8. It is proved in Thomson and Tait's Natural Philosophy, 6190 (7).
The scalar part is

\[ S \nabla \sigma = -\left( \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \right), \]

and the vector part is

\[ V \nabla \sigma = i \left( \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z} \right) + j \left( \frac{\partial Z}{\partial z} - \frac{\partial X}{\partial x} \right) + k \left( \frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} \right). \]

If the relation between \( X, Y, Z \) and \( \xi, \eta, \zeta \) is that given by equation (1) of the last theorem, we may write

\[ V \nabla \sigma = i \xi + j \eta + k \zeta. \]

See Theorem IV.

It appears therefore that the functions of \( X, Y, Z \) which occur in the two theorems are both obtained by the operation \( \nabla \) on the vector whose components are \( X, Y, Z \). The theorems themselves may be written

\[ \iiint S \nabla \sigma d\Sigma = \iiint S \cdot \sigma \mathbf{U} d\Sigma. \tag{III} \]

and

\[ \iiint S \sigma d\rho = -\iiint S \cdot \nabla \sigma \mathbf{U} d\Sigma; \tag{IV} \]

where \( d\Sigma \) is an element of a volume, \( d\Sigma \) of a surface, \( d\rho \) of a curve, and \( \mathbf{U} \) a unit-vector in the direction of the normal.

To understand the meaning of these functions of a vector, let us suppose that \( \sigma_0 \) is the value of \( \sigma \) at a point \( P \), and let us examine the value of \( \sigma - \sigma_0 \) in the neighbourhood of \( P \).

If we draw a closed surface round \( P \), then, if the surface-integral of \( \sigma \) over this surface is directed inwards, \( S \nabla \sigma \) will be positive, and the vector \( \sigma - \sigma_0 \) near the point \( P \) will be on the whole directed towards \( P \), as in the figure (1).

I propose therefore to call the scalar part of \( \nabla \sigma \) the "convergence" of \( \sigma \) at the point \( P \).

To interpret the vector part of \( \nabla \sigma \), let the direction of the vector whose components are \( \xi, \eta, \zeta \) be upwards from the paper and at right angles to it, and let us examine the vector \( \sigma - \sigma_0 \) near the point \( P \). It will appear as in the figure (2), this vector being arranged on the whole tangentially in the direction opposite to the hands of a watch.

I propose (with great diffidence) to call the vector part of \( \nabla \sigma \) the "rotation" of \( \sigma \) at the point \( P \).

In Fig. 3 we have an illustration of rotation combined with convergence.

Let us now consider the meaning of the equation

\[ V \nabla \sigma = 0. \]

This implies that \( \nabla \sigma \) is a scalar, or that the vector \( \sigma \) is the space-variation of some scalar function \( \psi \).

26.] One of the most remarkable properties of the operator \( \nabla \) is that when repeated it becomes

\[ \nabla^2 = -\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right), \]

an operator occurring in all parts of Physics, which we may refer to as Laplace's Operator.

This operator is itself essentially scalar. When it acts on a scalar function the result is scalar, when it acts on a vector function the result is a vector.

If, with any point \( P \) as centre, we draw a small sphere whose radius is \( r \), then if \( q \) is the value of \( q \) at the centre, and \( \bar{q} \) the mean value of \( q \) for all points within the sphere,

\[ q_0 - \bar{q} = \tau \nabla^2 q, \]

so that the value at the centre exceeds or falls short of the mean value according as \( \nabla^2 q \) is positive or negative.

I propose therefore to call \( \nabla^2 q \) the "concentration" of \( q \) at the point \( P \), because it indicates the excess of the value of \( q \) at that point over its mean value in the neighbourhood of the point.

If \( q \) is a scalar function, the method of finding its mean value is well known. If it is a vector function, we must find its mean value by the rules for integrating vector functions. The result of course is a vector.
A TREATISE ON

ELECTRICITY

AND MAGNETISM

By

JAMES CLERK MAXWELL

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612.] The volume-density of the free electricity at any point is found from the components of electric displacement by the equation

$$\rho = \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}. \quad (J)$$

613.] The surface-density of electricity is

$$\sigma = \ell f + mg + nh + \ell' f' + m'g' + n'h', \quad (K)$$

where \(\ell, m, n\) are the direction-cosines of the normal drawn from the surface into the medium in which \(f, g, h\) are the components of the displacement, and \(\ell', m', n'\) are those of the normal drawn from the surface into the medium in which they are \(f', g', h'.\)

614.] When the magnetization of the medium is entirely induced by the magnetic force acting on it, we may write the equation of induced magnetization,

$$\mathfrak{B} = \mu \mathfrak{H}, \quad (L)$$

where \(\mu\) is the coefficient of magnetic permeability, which may be considered a scalar quantity, or a linear and vector function operating on \(\mathfrak{H}\), according as the medium is isotropic or not.

615.] These may be regarded as the principal relations among the quantities we have been considering. They may be combined so as to eliminate some of these quantities, but our object at present is not to obtain compactness in the mathematical formulae, but to express every relation of which we have any knowledge. To eliminate a quantity which expresses a useful idea would be rather a loss than a gain in this stage of our enquiry.

There is one result, however, which we may obtain by combining equations (A) and (F), and which is of very great importance.

If we suppose that no magnets exist in the field except in the form of electric circuits, the distinction which we have hitherto maintained between the magnetic force and the magnetic induction vanishes, because it is only in magnetized matter that these quantities differ from each other.

According to Ampère’s hypothesis, which will be explained in Art. 833, the properties of what we call magnetized matter are due to molecular electric circuits, so that it is only when we regard the substance in large masses that our theory of magnetization is applicable, and if our mathematical methods are supposed capable of taking account of what goes on within the individual molecules, they will discover nothing but electric circuits, and we shall find the magnetic force and the magnetic induction everywhere identical. In order, however, to be able to make use of the electrostatic or the electromagnetic system of measurement at pleasure we shall retain the coefficient \(\mu\), remembering that its value is unity in the electromagnetic system.

616.] The components of the magnetic induction are by equations (A), Art. 591,

$$\mathfrak{a} = \frac{dH}{dy} - \frac{dG}{dz}, \quad (\mathfrak{a})$$

$$\mathfrak{b} = \frac{dF}{dz} - \frac{dH}{dx}, \quad (\mathfrak{b})$$

$$\mathfrak{c} = \frac{dG}{dx} - \frac{dF}{dy}. \quad (\mathfrak{c})$$

The components of the electric current are by equations (E), Art. 607, given by

$$4\pi u = \frac{dy}{dy} - \frac{d\beta}{dz}, \quad \text{magnetic force} \quad (\mathfrak{u})$$

$$4\pi v = \frac{dx}{dz} - \frac{d\gamma}{dx}, \quad (\mathfrak{v})$$

$$4\pi w = \frac{d\beta}{dx} - \frac{d\gamma}{dy}. \quad (\mathfrak{w})$$

According to our hypothesis, \(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\) are identical with \(\mu\alpha, \mu\beta, \mu\gamma\) respectively. We therefore obtain (when \(\mu\) is constant)

$$4\pi \mu u = \frac{d^2 G}{dx^2} + \frac{d^2 F}{dy^2} = \frac{d^2 c^2 G}{dx^2} + \frac{d^2 c^2 F}{dy^2}. \quad (1)$$

If we write

$$J = \frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz}, \quad (2)$$

and

$$\nabla^2 = -\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right), \quad (3)$$

we may write equation (1),

$$4\pi \mu u = \frac{dJ}{dx} + \nabla^2 F. \quad (4)$$

Similarly,

$$4\pi \mu v = \frac{dJ}{dy} + \nabla^2 G, \quad (5)$$

$$4\pi \mu w = \frac{dJ}{dz} + \nabla^2 H. \quad (6)$$

* The negative sign is employed here in order to make our expressions consistent with those in which Quaternions are employed.
This equation is true only if we take \( v \), \( w \), and \( \psi \) as the components of that electric flow which is due to the variation of electric displacement as well as to true conduction.

We have very little experimental evidence relating to the direct electromagnetic action of currents due to the variation of electric displacement in dielectrics, but the extreme difficulty of reconciling the laws of electromagnetism with the existence of electric currents which are not closed is one reason among many why we must admit the existence of transient currents due to the variation of displacement. Their importance will be seen when we come to the electromagnetic theory of light.

608.] We have now determined the relations of the principal quantities concerned in the phenomena discovered by Ørsted, Ampère, and Faraday. To connect these with the phenomena described in the former parts of this treatise, some additional relations are necessary.

When electromotive intensity acts on a material body, it produces in it two electrical effects, called by Faraday Induction and Conduction, the first being most conspicuous in dielectrics, and the second in conductors.

In this treatise, static electric induction is measured by what we have called the electric displacement, a directed quantity or vector which we have denoted by \( \mathcal{D} \), and its components by \( f, g, h \).

In isotropic substances, the displacement is in the same direction as the electromotive intensity which produces it, and is proportional to it, at least for small values of this intensity. This may be expressed by the equation

\[
\mathcal{D} = \frac{1}{4\pi} K \mathcal{E}, \quad (\text{Equation of Electric Displacement})
\]

where \( K \) is the dielectric capacity of the substance. See Art. 68.

In substances which are not isotropic, the components \( f, g, h \) of the electric displacement \( \mathcal{D} \) are linear functions of the components \( P, Q, R \) of the electromotive intensity \( \mathcal{E} \).

The form of the equations of electric displacement is similar to that of the equations of conduction as given in Art. 298.

These relations may be expressed by saying that \( K \) is, in isotropic bodies, a scalar quantity, but in other bodies it is a linear and vector function, operating on the vector \( \mathcal{E} \).

611.] The other effect of electromotive intensity is conduction. The laws of conduction as the result of electromotive intensity were established by Ohm, and are explained in the second part of this treatise, Art. 241. They may be summed up in the equation

\[
\mathcal{R} = C \mathcal{E}, \quad (\text{Equation of Conductivity})
\]

where \( C \) is the electromotive intensity at the point, \( \mathcal{R} \) is the density of the current of conduction, the components of which are \( p, q, \text{ and } r \), and \( C \) is the conductivity of the substance, which in the case of isotropic substances, is a simple scalar quantity, but in other substances becomes a linear and vector function operating on the vector \( \mathcal{E} \). The form of this function is given in Cartesian coordinates in Art. 298.

610.] One of the chief peculiarities of this treatise is the doctrine which it asserts, that the true electric current \( \mathcal{C} \), that on which the electromagnetic phenomena depend, is not the same thing as \( \mathcal{R} \), the current of conduction, but that the time-variation of \( \mathcal{D} \), the electric displacement, must be taken into account in estimating the total movement of electricity, so that we must write,

\[
\mathcal{C} = \mathcal{R} + \mathcal{D}, \quad (\text{Equation of True Currents})
\]

or, in terms of the components,

\[
\begin{align*}
\mathcal{C} &= p + \frac{d}{dt} \\
\mathcal{C} &= q + \frac{d}{dt} \\
\mathcal{C} &= r + \frac{d}{dt}
\end{align*}
\]

611.] Since both \( \mathcal{R} \) and \( \mathcal{D} \) depend on the electromotive intensity \( \mathcal{E} \), we may express the true current \( \mathcal{C} \) in terms of the electromotive intensity, thus

\[
\mathcal{C} = \left(C + \frac{1}{4\pi} K \frac{d}{dt}\right) \mathcal{E}, \quad (I)
\]

or, in the case in which \( C \) and \( K \) are constants,

\[
\begin{align*}
\mathcal{C} &= CP + \frac{1}{4\pi} K \frac{dP}{dt} \\
\mathcal{C} &= CQ + \frac{1}{4\pi} K \frac{dQ}{dt} \\
\mathcal{C} &= CR + \frac{1}{4\pi} K \frac{dR}{dt}
\end{align*}
\]

(Ia)
form, and position were properly adjusted, and that the magnet acts on the current in the same way as another current. These observations need not be supposed to be accompanied by actual measurements of the forces. They are not therefore to be considered as furnishing numerical data, but are useful only in suggesting questions for our consideration.

The question these observations suggest is, whether the magnetic field produced by electric currents, as it is similar to that produced by permanent magnets in many respects, resembles it also in being related to a potential?

The evidence that an electric circuit produces, in the space surrounding it, magnetic effects precisely the same as those produced by a magnetic shell bounded by the circuit, has been stated in Arts. 482-485.

We know that in the case of the magnetic shell there is a potential, which has a determinate value for all points outside the substance of the shell, but that the values of the potential at two neighbouring points, on opposite sides of the shell, differ by a finite quantity.

If the magnetic field in the neighbourhood of an electric current resembles that in the neighbourhood of a magnetic shell, the magnetic potential, as found by a line-integration of the magnetic force, will be the same for any two lines of integration, provided one of these lines can be transformed into the other by continuous motion without cutting the electric current.

If, however, one line of integration cannot be transformed into the other without cutting the current, the line-integral of the magnetic force along the one line will differ from that along the other by a quantity depending on the strength of the current. The magnetic potential due to an electric current is therefore a function having an infinite series of values with a common difference, the particular value depending on the course of the line of integration. Within the substance of the conductor, there is no such thing as a magnetic potential.

607.] Assuming that the magnetic action of a current has a magnetic potential of this kind, we proceed to express this result mathematically.

In the first place, the line-integral of the magnetic force round any closed curve is zero, provided the closed curve does not surround the electric current.

In the next place, if the current passes once, and only once, through the closed curve in the positive direction, the line-integral has a determinate value, which may be used as a measure of the strength of the current. For if the closed curve alters its form in any continuous manner without cutting the current, the line-integral will remain the same.

In electromagnetic measure, the line-integral of the magnetic force round a closed curve is numerically equal to the current through the closed curve multiplied by $4\pi$.

If we take for the closed curve the rectangle whose sides are $dy$ and $dz$, the line-integral of the magnetic force round the parallelogram is

$$ \left( \frac{\partial y}{\partial y} - \frac{\partial z}{\partial z} \right) dy dz,$$

and if $u, v, w$ are the components of the flow of electricity, the current through the parallelogram is $udy dz$.

Multiplying this by $4\pi$, and equating the result to the line-integral, we obtain the equation

$$ 4\pi u = \frac{\partial y}{\partial y} - \frac{\partial z}{\partial z}, $$

with the similar equations

$$ 4\pi v = \frac{\partial x}{\partial x} - \frac{\partial y}{\partial y}, $$

$$ 4\pi w = \frac{\partial z}{\partial z} - \frac{\partial x}{\partial x}, $$

(Equations of Electric Currents.)

which determine the magnitude and direction of the electric currents when the magnetic force at every point is given.

When there is no current, these equations are equivalent to the condition that

$$ ax + by + cz = D, $$

or that the magnetic force is derivable from a magnetic potential in all points of the field where there are no currents.

By differentiating the equations (E) with respect to $x, y,$ and $z$ respectively, and adding the results, we obtain the equation

$$ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, $$

which indicates that the current whose components are $u, v, w$ is subject to the condition of motion of an incompressible fluid, and that it must necessarily flow in closed circuits.