1) The current $j$ going through an anisotropic crystal is related to the electric field $E$ applied by Ohm’s Law:

$$j_i = \sigma_{ij} E_j, \quad (1)$$

where $\sigma_{ij}$ are the components of the conductivity tensor, given by

$$\begin{pmatrix}
1 & \sqrt{2} & 0 \\
\sqrt{2} & 3 & 1 \\
0 & 1 & 1
\end{pmatrix}$$

i) Show that there is one direction in the crystal along which no current can flow (5 points).

ii) Does the current flow equally easily in the two perpendicular directions to that in part i? If not, find the direction along which it flows most easily (5 points).

2) Given $A(x, y, z) = x^2 \hat{x} + y^2 \hat{y} + z^2 \hat{z}$, and let $S$ be the surface of a cube defined by $0 \leq x \leq 1$, $0 \leq y \leq 1$, and $0 \leq z \leq 1$.

i) Use Gauss theorem to calculate the flux of $A$ across $S$ (5 points).

ii) verify the answer obtained above by doing the surface integral directly (5 points).

3) Given the vector field $A = y \hat{x} + z \hat{y} + x \hat{z}$,

i) Check Stokes theorem for the surface given by the sphere $x^2 + y^2 + z^2 = 1$, with $z \geq 0$. Choose the normal to the surface that has positive $z$ component (5 points).

ii) Calculate the surface integral

$$\int_S (\nabla \times A) \cdot dS, \quad (2)$$

where $S$ is the surface given by the paraboloid $x^2 + y^2 + z = 1$, with $z \geq 0$ and the normal to the surface that has positive $z$ component (5 points).

4) Let $T$ be a tensor with components $T_{ij} = \delta_{ij} - \frac{x_i x_j}{r^2}$, where $r$ is the magnitude of the position vector, $x_i$ its components, and $i = 1, 2, 3$. Calculate

i) its trace (2 points).
ii) $T_{ij} x_i x_j$ (2 points).

iii) $\int T_{ij} \rho(r - r_0) d^3r$, for $\rho(r)$ given by a point charge (2 points).

iii) $\int T_{ij} \rho(r - r_0) d^3r$, for $\rho(r)$ given by an ideal dipole pointing in the $x$-direction (4 points).

5) Let the complex function $f(z)$ be analytic, $f = u + iv$ with $u, v$ real functions of $x, y$, and $z = x + iy$.

i) Show that if the real part of $f$ is constant, then $f$ must be constant (3 points).

ii) Show that if the absolute value of $f$ is a constant, then $f$ must be constant (7 points).