Third release of HyRec (May 2012): technical explanatory supplement

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This explanatory supplement provides detailed explanations of the modifications done in the third release of HyRec (May 2012): a more accurate numerical solution of the radiative transfer equations, and the inclusion of the explicit dependence of the recombination history on the fine structure constant and the electron mass.

I. IMPROVED NUMERICAL INTEGRATION

The modifications described below do not change the accuracy of the free-electron fraction in a significant way (at the level of a few 10^{-4} at most). However, they improve the accuracy of the Lyman-line distortion (which could be extracted from the code if desired), especially at early times when the radiation field is close to being thermal.

A. Improved post-Saha expansion at early phases of hydrogen recombination

At the highest redshifts, the ODE describing hydrogen recombination is stiff, and we follow the recombination history using perturbation theory around the Saha approximation, which we describe here. The free electron fraction is very close to the Saha equilibrium value: $x_e = x_e^{\rm S} + \Delta x_e$. We can therefore Taylor-expand the derivative of the free-electron fraction:

$$\dot{x}_e(x_e, z) \approx \dot{x}_e(x_e^{\rm S}, z) + \Delta x_e \frac{\partial \dot{x}_e}{\partial x_e}\Big|_{x_e^{\rm S}}. \tag{1}$$

On the other hand, we have

$$\dot{x}_e(x_e, z) \approx \frac{d(x_e^{\rm S})}{dt}. \tag{2}$$

This allows us to obtain an estimate of the departure from Saha equilibrium:

$$\Delta x_e \approx \left[\frac{d(x_e^{\rm S})}{dt} - \dot{x}_e(x_e^{\rm S}, z)\right] \Big/ \frac{\partial \dot{x}_e}{\partial x_e}\Big|_{x_e^{\rm S}}. \tag{3}$$

In the previous version of the code, it was assumed that $\dot{x_e}(x_e^{\rm S}, z) = 0$. In fact, the rate of change of the free electron fraction $\dot{x}_e(x_e, z, \Delta f_\nu)$ also depends on the photon distortion $\Delta f_\nu(z; \Delta x_e(z' > z))$, which itself depends on the free electron fraction at earlier times. Therefore, at a given time z, and for $\Delta f_\nu \neq 0$, the function $\dot{x}_e$ evaluated at $x_e^{\rm S}$ does not vanish in general, and it is explicitly evaluated in the new version of the code, making the post-Saha expansion more accurate.

We use the post-Saha expansion until the departure from Saha equilibrium reaches $\Delta x_e = 3 \times 10^{-4}$, after which we switch to the numerical integration of the recombination ODE. We checked that the error during the post-Saha phase is at most $|\Delta x_e| \sim 5 \times 10^{-5}$ by comparing against a much finer time-step computation where one can avoid the post-Saha expansion.

B. Numerical radiative transfer equations in terms of departures from equilibrium

The numerical radiative transfer equations solved in the previous version of the code involved the absolute excited level populations x_i and photon occupation number f_ν or equivalently the virtual level populations $x_b \equiv x_{1s} f_{\nu_b}$. At the beginning of hydrogen recombination, the excited states are very close to Boltzman equilibrium with the ground state, and the radiation field is nearly thermal. The net rate of decays to the ground state therefore depends on nearly cancelling terms. A better accuracy can be reached by rewriting the radiative transfer equations in terms of departures from thermal equilibrium with the ground state, using the variables

$$\Delta x_i \equiv x_i - x_{1s}\frac{g_i}{g_{1s}} {\rm e}^{-E_{i1}/T_{\rm r}}, \quad \Delta x_b \equiv x_b - x_{1s}{\rm e}^{-h\nu_b/T_{\rm r}} \equiv x_{1s}\Delta f_{\nu_b}, \quad \Delta(x_e^2) \equiv x_e^2 - \frac{(x_e^S)^2}{1-x_e^S} x_{1s} \equiv x_e^2 - s(T_r) x_{1s}, \tag{4}$$

where

$$s(T_r) \equiv \frac{(2\pi\mu_e T_r)^{3/2}}{h^3 n_{\rm H}} {\rm e}^{-E_I/T_r} = \frac{(x_e^{\rm S})^2}{1-x_e^{\rm S}}. \tag{5}$$

We also use the departures of the effective recombination coefficients from their values if matter and radiation had the same temperature T_r,

$$\Delta\mathcal{A}_i \equiv \mathcal{A}_i(T_{\rm m}, T_{\rm r}) - \mathcal{A}_i(T_{\rm m} = T_{\rm r}, T_{\rm r}). \tag{6}$$

With these new variables, several terms cancel out of the recombination equation, and we are left with the following new equations:

• The discretized radiative transfer equation (Eqs. (86), (84) and (81) of Ref. [1]) becomes

$$T_{b,2s}\Delta x_{2s} + T_{b,2s}\Delta x_{2p} + \sum_{b'=b-1}^{b+1} T_{b,b'}\Delta x_b = \Pi_b x_{1s}\Delta f_{\nu_b+\epsilon} T_{b,b}. \tag{7}$$

• The rate equation for each state $i = 2s, 2p$ (Eqs. (93), (92) of Ref. [1]) becomes

$$\sum_{j=2s,2p} T_{i,j}\Delta x_j + \sum_b T_{i,b}\Delta x_b = \Delta S_i, \quad \text{where} \tag{8}$$

$$\Delta S_i \equiv n_{\rm H}\left[s x_{1s}\Delta\mathcal{A}_i + \mathcal{A}_i\Delta(x_e)^2\right] + 3\sum_{n\geq 2} R_{\rm Lyn} P^i_{np}\Delta f^+_{np}. \tag{9}$$

• Finally, the rate of change of the free electron fraction (Eq. (39) of Ref. [1]) becomes

$$\dot{x}_e = -\sum_{i=2s,2p}\left\{ n_{\rm H}\left[s x_{1s}\Delta\mathcal{A}_i + \mathcal{A}_i\Delta(x_e)^2\right] - \Delta x_i \mathcal{B}_i \right\}. \tag{10}$$

As a result of this improved numerical treatment, the user can extract an accurate distortion field, as is shown in Fig. 1. Note that distortion photons from Helium recombination are not included (but they are followed in our analytic treatment of Helium recombination), and neither are photons emitted by neutral hydrogen while helium recombines (see Ref. [2]).

II. DEPENDENCE OF THE RECOMBINATION HISTORY ON FUNDAMENTAL CONSTANTS

Various theories can result in the variations of some fundamental constants (for a review, see e.g. Ref. [3]), and the CMB can and has been used to test for the variations of the fine structure constant $\alpha_{\rm fs}$ and the electron mass[1] m_e, which affect the rate of recombination as well as the scattering rate by free electrons. Here we assume that the fine-structure constant and the electron mass may have a different value at recombination than today (and for the purpose of simplicity we assume these values to be constant during the whole recombination process).

We summarize the dependencies below:

- *Transition frequencies:*

$$\nu \propto \alpha_{\rm fs}^2 m_e \tag{11}$$

- *Einstein-A coefficients (single photon electric dipole transitions):*

$$A \propto \alpha_{\rm fs}^5 m_e \tag{12}$$

- *Two-photon decay rate:*

$$\frac{d\Lambda}{d\nu} \propto \alpha_{\rm fs}^6, \tag{13}$$

$$\Lambda \propto \alpha_{\rm fs}^8 m_e \tag{14}$$

[1] To be exact, all quantities related to the hydrogen atom depend on the reduced mass of the electron-proton system, $\mu_e = m_e/(1+m_e/m_p)$ rather than m_e. However, $\Delta\mu_e/\mu_e = \Delta m_e/m_e\,[1+\mathcal{O}(\Delta m_e/m_p)]$, and therefore we assume that $\Delta\mu_e/\mu_e = \Delta m_e/m_e$. Note however that HyRec uses the correct reduced mass for all zeroth order quantities.

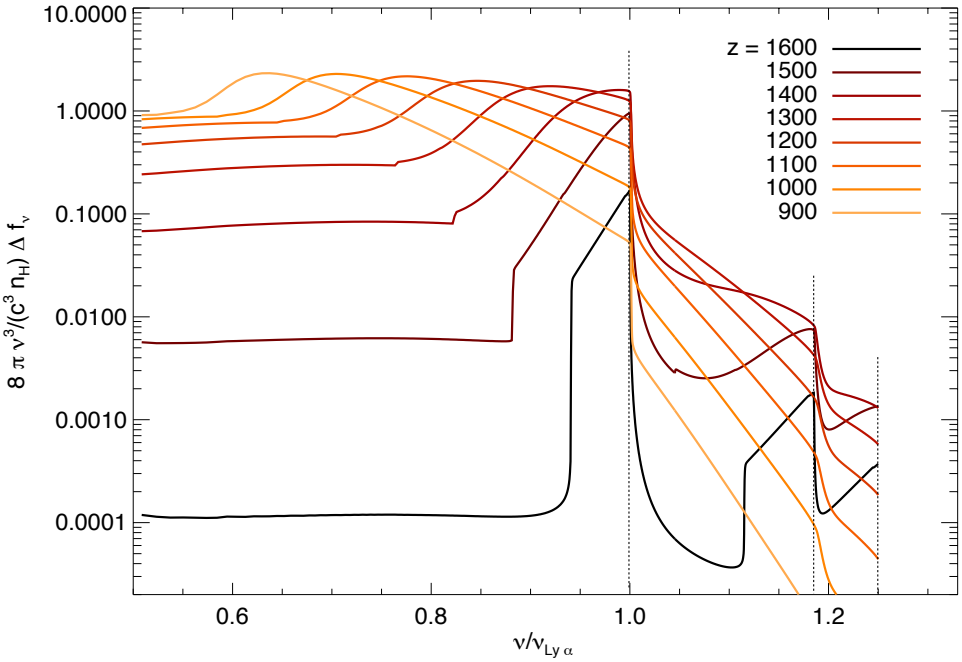

FIG. 1. Number of Lyman-distortion photons per hydrogen atom per log-frequency interval, as a function of redshift. The three vertical dashed lines show the position of the Ly-α, Ly-β and Ly-γ lines. The sudden drops on the red side of Ly-α and Ly-β at early times are unphysical and arise because we only start following hydrogen recombination (hence the spectrum) at $z = 1700$. Note that distortion photons emitted during helium recombination, both by helium and hydrogen (see for example Ref. [2]), are not accounted for, and would be dominant redward of these discontinuities at early times.

- *Lyman-α escape rate:*

$$R_{\mathrm{Ly}\alpha} \propto \nu_{\mathrm{Ly}\alpha}^3 \propto \alpha_{\mathrm{fs}}^6 m_e^3. \tag{15}$$

- *Recombination coefficients and photoionization rates:*

$$\alpha = \frac{\alpha_{\mathrm{fs}}^2}{m_e^2}\, F\left[\frac{T_m}{\alpha_{\mathrm{fs}}^2 m_e}, \frac{T_r}{\alpha_{\mathrm{fs}}^2 m_e}\right], \tag{16}$$

$$\beta = \alpha_{\mathrm{fs}}^5 m_e\, G\left[\frac{T_r}{\alpha_{\mathrm{fs}}^2 m_e}\right], \tag{17}$$

where F and G are some functions. Since the prefactors in β are the same as those of the Einstein-A coefficients, the probabilities P_{nl}^{2l} defined in Ref. [4] are only functions of $T_r/(\alpha_{\mathrm{fs}}^2 m_e)$, without any prefactors. The dependencies given above therefore carry over to the effective recombination and photionization rates to $2s$ and $2p$ defined in Ref. [4] and used in HyRec. The effective $2p \to 2s$ transition rate has the same prefactor as the Einstein-A coefficient.

- *Photoionization cross-section from the ground state of hydrogen* (required for helium recombination):

$$\sigma_{pi} \propto \alpha_{\mathrm{fs}} a_0^2 \propto \frac{1}{\alpha_{\mathrm{fs}} m_e^2}, \tag{18}$$

- *Thomson cross-section:*

$$\sigma_T \propto \frac{\alpha_{\mathrm{fs}}^2}{m_e^2} \tag{19}$$

Note that the Thomson cross-section is used in HyRec only to compute the matter temperature evolution. Obviously, it should also be modified in the Boltzmann code calling HyRec when computing CMB anisotropies.

We show the impact of changes in α_{fs} and m_e on the recombination history and the visibility function in Fig. 2.

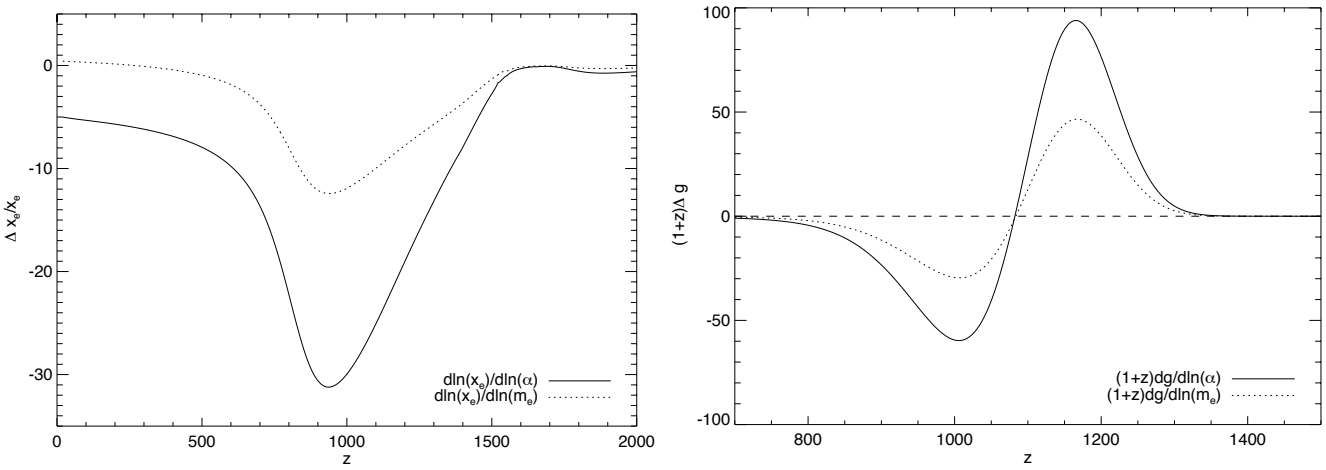

FIG. 2. Changes to the recombination history (left) and the visibility function (right) due to a variation of the fine-structure constant or the electron mass.

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