General Relativity Fall 2017
Homework 1

due September 12th 2017

Exercise 1: Change of bases

Given two bases $e_{(\mu)}$ and $e_{(\mu')}$, related through $e_{(\mu')} = \Lambda^{\mu}{}_{\mu'} e_{(\mu)}$, show that their dual bases $e^{(\mu)}$ and $e^{(\mu')}$ are related through $e^{(\mu)} = \Lambda^{\mu}{}_{\mu'} e^{(\mu')}$. [If the notation displeases you, feel free to use a better one]. Explicitly derive the rules of transformations of the components of a tensor of rank $(1, 1)$.

Exercise 2: Orthonormal bases

Given a vector space of dimension n and a metric tensor on that space [i.e. a symmetric, non-degenerate tensor of rank (0, 2)], prove that there exist bases in which $g_{\mu\nu} = \pm\delta_{\mu\nu}$.
Hint: work with the quadratic form $Q(X^1, ..., X^n) \equiv g_{\mu\nu} X^\mu X^\nu$ and proceed recursively. First consider the case where one of the diagonal components does not vanish, say g_{11}, and find a new coordinate $\tilde{X}^1$ such that $Q(X^1, ..., X^n) = \pm(\tilde{X}^1)^2 + \tilde{Q}(X^2, ..., X^n)$. Second, consider the case where all the diagonal components vanish, and suppose $g_{12} \neq 0$. Find new coordinates $\tilde{X}^1, \tilde{X}^2$ such that $Q(X^1, ..., X^n) = (\tilde{X}^1)^2 - (\tilde{X}^2)^2 + \tilde{Q}(X^3, ..., X^n)$. Relate these new coordinates to new bases.

Exercise 3: Sylvester's law of inertia

Suppose we have two orthonormal bases $e_{(\mu)}$ and $f_{(\nu)}$ of a n-dimensional vector space $\mathcal{V}$, in which the metric $\mathbf{g}$ has signature [i.e. (number of -1's, number of +1's)] $(p, n-p)$ and $(q, n-q)$, respectively. Show that $p = q$.
Hint: Proceed by contradiction: if $p < q$, define the linear map $L : \mathcal{V} \to \mathbb{R}^{n-q+p}$ such that

$$L(V) \equiv (\mathbf{g}(V, e_{(1)}), ..., \mathbf{g}(V, e_{(p)}), \mathbf{g}(V, f_{(q+1)}), ..., \mathbf{g}(V, f_{(n)})).$$

Argue that there must exist a non-zero vector V_0 for which $L(V_0) = 0$. Compute the norm of this vector in two ways and show that it is positive and negative, hence zero. Then show that $V_0 = 0$, leading to a contradiction.

Exercise 4: The tangent space as the space of directional derivative operators

We denote by $\mathcal{F}$ the set of infinitely differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$. We define the tangent space at a point $P \in \mathbb{R}^n$ as the space of linear operators $V : \mathcal{F} \to \mathbb{R}$ that satisfy Leibniz's rule, $V(fg) = f(P)V(g) + g(P)V(f)$. Show that (i) these operators acting on constant functions give zero (ii) any such operator can be written as a linear combination of partial derivative operators, $V = V^\mu \partial_\mu$.
Hint: given $f \in \mathcal{F}$, show that at any point P with coordinates x_P^μ, there exists n differentiable functions $H_\mu(x)$ s.t.

$$f(x) = f(x_P) + (x^\mu - x_P^\mu) H_\mu(x). \tag{1}$$

Prove (ii) using (i) and Leibniz rule applied to this expression.

Exercise 5: Levi-Civita tensor in spherical coordinates

Compute the all-covariant and all-contravariant components of the Levi-Civita tensor in the spherical polar coordinate basis of $\mathbb{R}^3$.

Exercise 6: index manipulation fun

(i) If the tensor $T_{\alpha\beta}$ is symmetric, show that $T^\alpha{}_\beta = T_\beta{}^\alpha$.
(ii) Write explicitly the antisymmetric part $T_{[\alpha\beta\gamma]}$ of the rank $(0, 3)$ tensor $T_{\alpha\beta\gamma}$.
(iii) Given a rank (0,2) tensor $T_{\alpha\beta}$, what is the rank of the tensor $T_{\alpha\beta} T_\gamma{}^\sigma T^{\beta\gamma}$? How about $T_{\alpha\beta} T_\gamma{}^\alpha T^{\beta\gamma}$?
(iv) Suppose that $n = 4$ and that, in a given basis where $g_{\mu\nu} = \eta_{\mu\nu}$, the components of the tensor $T_{\alpha\beta}$ are given by

$$T_{\mu\nu} = \begin{pmatrix} 2 & 0 & -1 & 1 \\ 3 & 0 & 2 & 1 \\ -2 & 1 & 0 & -1 \\ -1 & 0 & -2 & 3 \end{pmatrix}. \tag{2}$$

Compute $T_{\alpha(\beta} T_{\gamma)}{}^\alpha T^{[\beta\gamma]}$. *Hint*: think a little bit before starting!!