General Relativity Fall 2017
Homework 2

due September 19th 2017

Exercise 1: straight line trajectories as the path of maximum proper time

Consider two spacetime points P_1, P_2 with a timelike separation, and paths joining the two such that the tangent vector to the path is everywhere timelike or null. We showed in class that the proper time is extremized for the trajectory with constant 4-velocity. Show that this trajectory *maximizes* proper time.
Hint: Work in the rest-frame of a particle traveling at constant velocity on the trajectory of extremal proper time, and rescale the time so the proper time elapsed along this trajectory is $\tau_0 = 1$. Write explicity the integrand of τ in terms of $\delta\dot{t} = d\delta t/\delta\lambda$ and $\delta\vec{x} = d\delta\vec{x}/\delta\lambda$, where $\lambda \in [0,1]$ parametrizes the trajectory. Taylor-expand to second-order in small deviations.

Exercise 2: ideal fluid

(i) We recall that the Lorentz transformation for a Lorentz boost with velocity v along the x direction takes the form

$$\Lambda = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (1)$$

Write down explicitly the transformation of the components of the stress-energy-momentum tensor under a Lorentz boost.
(ii) Starting from the stress-energy-momentum tensor of an ideal fluid in the fluid rest frame, $\tilde{T}^{00} = \rho$, $\tilde{T}^{0i} = 0$, $\tilde{T}^{ij} = P\delta^{ij}$, transform to a general frame in which the fluid is moving with 3-velocity $\vec{v}$ to derive the general expression $T^{\mu\nu} = \rho U^\mu U^\nu + P(\eta^{\mu\nu} + U^\mu U^\nu)$, where U^μ is the fluid's 4-velocity.
(iii) Starting from the conservation of the energy-momentum tensor, write explictly the relativistic ideal fluid equations in terms of the fluid's proper density ρ and 3-velocity $\vec{v}$.
(iv) We assume that the ideal fluid is made of identical particles whose number is conserved. We define the particle number-current 4-vector $N^\alpha = nU^\alpha$, where n is the fluid's proper number density and U^α the fluid's velocity. Combining the conservation of particle number $\partial_\alpha N^\alpha$ with the conservation of energy-momentum, show that the specific entropy σ, defined through the differential relation $Td\sigma \equiv Pd(1/n) + d(\rho/n)$, is constant for any point that moves along the fluid. [*Hint*: a quantity X is constant for a point moving along the fluid if $\partial_t X + \vec{v}\cdot\partial_{\vec{x}} X = 0$].

Exercise 3: Electromagnetic stress-energy-momentum tensor

The electromagnetic stress-energy-momentum tensor is defined as $T^{\mu\nu}_{\rm em} = F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}\eta^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}$. (i) Show that $\partial_\mu T^{\mu\nu}_{\rm em} = -F^{\nu\gamma}J_\gamma$, where J^μ is the charge current. (ii) Write explicitly $T^{00}_{\rm em}$, $T^{0i}_{\rm em}$ and $T^{ij}_{\rm em}$ in terms of the electric and magnetic fields, defined by $F^{0i} = E^i$, $\tilde{\epsilon}^{ijk}B_k = F^{ij}$, where $\tilde{\epsilon}^{ijk}$ is the Levi-Civita symbol (= +1 if ijk is an even permutation of (1,2,3), -1 for odd permutations, 0 otherwise).

Exercise 4: Thomas precession

(i) Consider a massive particle with 4-velocity u^α. If S^α is a 4-vector defined along the particle's worldline, explain in words what the condition $u_\alpha S^\alpha = 0$ implies [I am looking for a sentence that contains "in the particle's rest frame"]. How about the conditions $(\eta_{\alpha\beta} + u_\alpha u_\beta)S^\beta = 0$?
(ii) If $a^\alpha = du^\alpha/d\tau$ is a massive particle's 4-acceleration, compute $u^\alpha a_\alpha$ [*hint*: it's a simple answer].
(iii) The spin of a particle can be represented by a spacelike 4-vector S^α orthogonal to u_α and with constant norm $(S^\alpha S_\alpha)^{1/2}$. Assuming there are no torques acting on the particle, show that the rate of change of S^α along the particle's worldline is given by $dS^\alpha/d\tau = (S^\beta a_\beta)u^\alpha$ [*hint*: argue that this Lorentz-vector can only be constructed out of contractions of u^α and a^α; then show that these contractions must take the form $dS^\alpha/d\tau = Au^\alpha + Ba^\alpha$; find A and B from the requirement that S^α remains orthogonal to u^α and keeps a constant norm.]