General Relativity Fall 2017
Homework 3

due September 26th 2017

Exercise 1: The difference between two covariant derivatives is a tensor

The goal of this problem is to prove the statement made in class that the difference between two covariant derivatives is a tensor field. We recall that a covariant derivative gives the usual gradient when acting on scalars (hence, all covariant derivatives agree on scalars), and satisfies Leibniz' rule.

(i) Consider two covariant derivatives ∇ and $\tilde{\nabla}$, a vector field V^α and a scalar field (i.e. a function) f. Show that

$$\tilde{\nabla}_\beta(fV^\alpha) - \nabla_\beta(fV^\alpha) = f\left(\tilde{\nabla}_\beta V^\alpha - \nabla_\beta V^\alpha\right). \tag{1}$$

(ii) Using (i), show that $\tilde{\nabla}_\beta(V^\alpha) - \nabla_\beta(V^\alpha)\big|_p$ only depends on the value of V^α at p – in contrast, $\nabla_\beta V^\alpha$ and $\tilde{\nabla}_\beta V^\alpha$ depend a priori on the values of V^α on a neighborhood of p.

Hint: Pick a coordinate system in the neighborhood of p and the associated coordinate basis $\partial_{(\mu)}$ [to be clear, for each $\mu = 1, ..., n$, $\partial_{(\mu)}$ is a vector *field*, defined over some neighborhood of p], and decompose V^α on this basis.

(iii) Argue that $\tilde{\nabla}_\beta(V^\alpha) - \nabla_\beta(V^\alpha)\big|_p = C^\alpha{}_{\beta\gamma}V^\gamma\big|_p$, where $C^\alpha{}_{\beta\gamma}$ is a tensor of rank $(1,2)$. I am looking for a simple but well-formulated argument that involves the definition of a tensor, as a linear map from vectors and dual vectors to real numbers. Make sure to clearly specify what quantities are *fields* and what quantities are vectors/tensors at a specific point.

Exercise 2: Connection coefficients for the metric-compatible, torsion-free covariant derivative

(i) Starting with $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda}V^\lambda$, derive the expression of $\nabla_\mu g_{\nu\lambda}$ in terms of Christoffel symbols.

(ii) Requiring this tensor to vanish (i.e. the covariant derivative to be metric-compatible), derive the expression for the Christoffel symbols $\Gamma^\nu_{\mu\lambda}$.

Exercise 3: Parallel transport using embedding

The expression above relied on a completely intrinsic definition of manifolds, tangent spaces, etc... Here we see how it is consistent with the intuitive notion of parallel transport in an embedding flat space.

Consider a smooth n-dimensional surface $\mathcal{M}$ of a flat N-dimensional space $\mathcal{S}$, with $N > n$. We denote by $\mathcal{V}_\mathcal{M}\big|_x$ the tangent space to the surface at a point $x \in \mathcal{M}$ and by $\mathcal{V}_\mathcal{S}\big|_y$ the tangent space to the flat space at a point $y \in \mathcal{S}$. Note that for all $x \in \mathcal{M}$, $\mathcal{V}_\mathcal{M}\big|_x$ is a subspace of $\mathcal{V}_\mathcal{S}\big|_x$. Because the space $\mathcal{S}$ is flat, we can lay out a global system of rectilinear coordinates y^a, $a = 1, ..., N$. In particular, this allows us to unambiguously parallel transport a vector V from $\mathcal{V}_\mathcal{S}\big|_y$ to $\mathcal{V}_\mathcal{S}\big|_{y'}$, by simply requiring that the coordinates V^a are constant – just like you would intuitively "parallel transport" an arrow in flat 3-D space. We denote by $e_{(a)} \equiv \partial/\partial y^a$ the basis vectors of this tangent space - they are in principle vector fields, but since the coordinates are rectilinear, they are effectively constant.

We denote by x^μ, $\mu = 1, ..., n$ coordinates on $\mathcal{M}$, and associated coordinate basis $\partial_{(\mu)} = \partial y^a/\partial x^\mu \; e_{(a)}$. These are vector *fields*, which vary from one point to another, and belong to $\mathcal{V}_\mathcal{M}$. We denote by $g_{\mu\nu}$ the components of the metric on the surface and h_{ab} the *constant* metric on the embedding space. We suppose moreover that the embedding is *isometric*, i.e. that $\delta s^2 = g_{\mu\nu}\delta x^\mu \delta x^\nu = h_{ab}\delta y^a \delta y^b$, if δy^a is the infinitesimal change in y^a corresponding to δx^μ.

(i) Show that

$$g_{\mu\nu} = h_{ab}\frac{\partial y^a}{\partial x^\mu}\frac{\partial y^b}{\partial x^\nu}. \tag{2}$$

(ii) Consider a vector $A^a \in \mathcal{V}_\mathcal{S}\big|_x$, where $x \in \mathcal{M}$. Decompose this vector as a part tangent to the surface and a part normal to it:

$$A = A_{||} + A_{\perp}, \tag{3}$$

$$A_{||} = A^{\mu}_{||}\partial_{(\mu)} \in \mathcal{V}_{\mathcal{M}}\big|_{x}, \tag{4}$$

$$h_{ab}A^{a}_{\perp}\partial^{b}_{(\mu)} = 0, \quad \forall\mu. \tag{5}$$

Show that

$$A^{\mu}_{||} = g^{\mu\nu}h_{ab}A^{a}\frac{\partial y^{b}}{\partial x^{\nu}}. \tag{6}$$

(iii) Consider a vector A that, at a point x of the manifold, lies in the tangent space of the manifold, i.e. $A_{\perp}(x) = 0$. Define the vector $A(x+\delta x)$, whose components $A^{a}(x+\delta x) = A^{a}(x)$, i.e. $A(x+\delta x)$ is $A(x)$ parallel-transported to $x+\delta x$ in the intuitive way in a flat space. Compute $A^{\mu}_{||}(x+\delta x)$ at first order in δx (be careful to keep track of what is constant and not!). Show that

$$\frac{\partial A^{\mu}_{||}}{\partial x^{\nu}} = -\Gamma^{\mu}{}_{\nu\lambda}A^{\lambda}_{||}, \tag{7}$$

where $\Gamma^{\mu}{}_{\nu\lambda}$ is the Christoffel symbol previously defined.