General Relativity Fall 2017
Homework 5

due October 10th 2017

Exercise 1: Curvature of a sphere

Consider a sphere with radius r. The metric on this sphere is $ds^2 = r^2(d\theta^2 + \sin^2\theta d\varphi^2)$. Compute the Ricci scalar of this metric. You can use the results of the previous howework: the only non-vanishing Christoffel symbols are $\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta$ and $\Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \cos\theta/\sin\theta$ [the r^2 in the metric cancels out the r^{-2} in the inverse metric].

Exercise 2: Rindler spacetime

Consider a spacetime with two-dimensional metric

$$ds^2 = -(a\ x)^2 dt^2 + dx^2, \tag{1}$$

where the parameter a is a constant, and the coordinates (t, x) are defined over $-\infty < t < \infty$ and $0 < x < \infty$.

(i) Compute the connection coefficients of this metric with your favorite method.
(ii) Compute the Ricci scalar of this metric. What does your result imply? Make sure to invoque the Riemann tensor in your argumentation.
(iii) What is the acceleration 2-vector of observers of constant x?
(iv) Show that null geodesics are such that $at = \pm\ln(a\ x)+$ constant.
(v) Define the coordinates $u \equiv at - \ln(a\ x)$ and $v \equiv at + \ln(a\ x)$. What are the null geodesics in these coordinates? What are the metric coefficients in these coordinates?
(vi) Can you find new coordinates T, X in which $ds^2 = -dT^2 + dX^2$? [*Hint*: start by finding coordinates U, V in which $ds^2 = -dUdV$]. Give an explicit expression of the old coordinates in terms of the new coordinates.
(vii) In a spacetime diagram in the (T, X) coordinates, draw the worldlines of observers of constant x, and those of observers of constant t.

Exercise 3: Geometric units

Geometric units are units in which $G = c = 1$. In those units, masses, lengths and times have the same dimensions.

(i) Give the mass of the Earth in centimeters. Explain how to get this by writing explicitly G's and c's.
(ii) What is the Newtonian gravitational potential Φ at the surface of the Earth (in the simplest unit)?
(iii) We saw that $R_{0i0j} \approx \partial_i\partial_j\Phi$ in the Newtonian limit. Assuming the Ricci scalar $R \sim R_{0i0j}$ and drawing an analogy with Exercise 1, what is the characteristic radius of curvature of spacetime at the surface of the Earth? How about at the surface of the Sun?
(iv) The reduced Planck constant $\hbar \approx 10^{-27}$ g cm^2 s^{-1}. In geometric units, what is the value of $\hbar$ in grams? In centimeters? In seconds? These are the Planck mass, length, and time, respectively.

Exercise 4: minimal substitution subtleties

In curved spacetime Maxwell's equations are

$$\nabla^\alpha F_{\alpha\beta} = -J_\beta, \tag{2}$$
$$\nabla_{[\alpha}F_{\beta\delta]} = 0. \tag{3}$$

The second equation implies that one can find a vector potential A^α such that $F_{\alpha\beta} = \nabla_{[\alpha}A_{\beta]}$.
What is the equation satisfied by the vector potential in the Lorenz gauge $\nabla_\alpha A^\alpha = 0$?