General Relativity Fall 2017
Homework 6

due October 17th 2017

Exercise 1: Killing fields and symmetries

Here you will derive Killing's equation from a different route than that taken in class.

Consider a general (i.e. NOT nearly flat) spacetime with metric $g_{\mu\nu}$, and consider an infinitesimal change of coordinates $y^\mu = x^\mu - \xi^\mu$, with $|\xi^\mu_{\ ,\lambda}| \ll 1$ *and* ξ^μ much smaller than the characteristic lengthscale over which $g_{\mu\nu}$ changes i.e. $|\xi^\sigma| \ll |g_{\mu\nu}/\partial_\sigma g_{\mu\nu}|$.

(i) To linear order in ξ, compute the components $g^{(y)}_{\mu\nu}$ in the y-coordinate system as a function of $g^{(x)}_{\mu\nu}$, at the same spacetime event. This is the usual, local coordinate transformation we are used to.

(ii) Now express $g^{(y)}_{\mu\nu}$ at the event of y-coordinates $y^\lambda = z^\lambda$ as a function of $g^{(x)}_{\mu\nu}$ at the event of x-coordinates $x^\lambda = z^\lambda$. These are *not* the same spacetime events.

(iii) Show that $\Delta g_{\mu\nu} \equiv g^{(y)}_{\mu\nu}(y^\lambda = z^\lambda) - g^{(x)}_{\mu\nu}(x^\lambda = z^\lambda) = \xi_{\mu;\nu} + \xi_{\nu;\mu} = 2\xi_{(\mu;\nu)}$, where $\xi_\mu \equiv g_{\mu\nu}\xi^\nu$. This implies that $\Delta g_{\mu\nu} = 0$ if and only if ξ^μ satisfies Killing's equation $\xi_{(\mu;\nu)} = 0$: a Killing field describes a symmetry of the spacetime.

Exercise 2: constraint equations

Using the contracted Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$, show that the 0μ Einstein field equations do not contain any second time derivatives: these are pure constraint equations. [Hint: rewrite $\nabla^\mu G_{\mu\nu}$ in the form $\partial^0 G_{0\nu} = S_\nu$, and argue that S_ν contains at most two time derivatives].

Exercise 3: weak-field Einstein equation

In class we will derive the form of the linearized Einstein tensor, up to an overall normalization constant:

$$G_{\mu\nu} = A\left(\Box h_{\mu\nu} - 2\,\partial^\alpha\partial_{(\mu}h_{\nu)\alpha} + \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h\right). \tag{1}$$

Compute the coefficient A explicitly, by picking as simple a metric as possible – for instance, suppose only $h_{00} \neq 0$, and find A by computing both sides.

Exercise 4: gauge-invariance of the Riemann tensor at linear order

Consider a nearly flat spacetime, and a globally nearly-Minkowski coordinate system, so that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $|h_{\mu\nu}| = \mathcal{O}(\epsilon) \ll 1$. In class we gave a simple argument for why the components of the Riemann tensor are invariant under gauge transformations at linear order. Here we will show this again, in a more explicit way.

(i) Derive the all-covariant components of the Riemann tensor $(R_{\mu\nu\rho\sigma})$ to linear order in $h_{\mu\nu}$. Beware that this is *not* a locally inertial coordinate system. Explain clearly every step.

(ii) Compute the change of these components under a gauge transformation $h_{\mu\nu} \to h_{\mu\nu} + 2\xi_{(\mu,\nu)}$.