General Relativity Fall 2017
Homework 9

due Novermber 14th 2017

Exercise 1: order-of-magnitude estimates

(i) Suppose you wave your arms at a frequency of 1Hz. Estimate the wavelength of gravitational waves that you radiate, the characteristic amplitude of the gravitational wave strain (i.e. of the components of $h_{ij}^{\rm TT}$) one wavelength away and the power radiated in gravitational waves. Suppose that the gravitational-wave energy is carried by individual gravitons of energy $h_{\rm P}\nu$, where $h_{\rm P}$ is Planck's constant and ν is the frequency. How long does it take to radiate one single graviton?

(ii) Consider an equal-mass binary with total mass $M = m\ M_\odot$ (i.e. m is the mass in solar mass units), on a circular orbit. Suppose the binary is at a distance $d = 100\ d_{100}$ Mpc away (i.e. d_{100} is the distance in units of 100 Mpc). Estimate the characteristic frequency and strain of gravitational waves when the semimajor axis is $a \approx 10M$ – express your result as a function of m and d_{100}. Suppose a GW detector has a sensitivity to strains as low as 10^{-21} for frequencies in the range $10^2 - 10^3$ Hz (this is, very roughly, the case for LIGO). Using your estimates above (i.e. for $a \approx 10M$), estimate the characteristic masses to which the detector is sensitive. As a function of mass, what is the maximum distance at which a binary can be detected?

Exercise 2: the effective stress-energy tensor in the quasi-Newtonian limit

Consider the quasi-Newtonian metric

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)\delta_{ij}dx^i dx^j, \tag{1}$$

with $|\Phi| \sim v^2$, where $v \ll 1$ is the characteristic velocity inside the source.

(i) Compute the Einstein tensor of this metric to quadratic order in metric perturbations. From this extract the pseudo-tensor $t^{\mu\nu} = -\frac{1}{8\pi}\left(G^{\mu\nu} - G^{\mu\nu}_{(1)}\right)$, where $G^{\mu\nu}_{(1)}$ is the part of $G^{\mu\nu}$ linear in metric perturbations. Assume that, moreover, $\partial_t\Phi \sim v\partial_i\Phi$, and keep only the dominant terms.

(ii) Suppose the source of the gravitational field is an ideal fluid with

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}, \tag{2}$$

with $p/\rho \sim v^2 \ll 1$. Define $T^{\mu\nu}_{\rm eff} \equiv T^{\mu\nu} + t^{\mu\nu}$. Compute $\partial_\mu T^{\mu 0}_{\rm eff}$ and $\partial_\mu T^{\mu i}_{\rm eff}$ to lowest order in v (the order of the dominant term will be different for the two components). Simplify using Poisson's equation $\partial_i\partial_i\Phi = 4\pi\rho$. You should find that to lowest order, $\partial_\mu T^{\mu\nu}_{\rm eff} = 0$ reduce to the Newtonian hydrodynamic equations, which can also be obtained from $\nabla_\mu T^{\mu\nu} = 0$.