General Relativity Fall 2017
Midterm exam

due October 24th

Requirements:

• You can take as long as you want, but please work on this midterm **on your own**.

• Once you start, you **may not** use any textbook or reference **except** lecture notes, past homeworks and solutions, and Carroll's textbook or arXiv lecture notes. Googling is not allowed.

• Please bring **2 copies** of your work to my office on October 24th.

A few useful expressions:

• The electromagnetic tensor $F_{\alpha\beta}$ can be written in terms of a vector potential A_α:

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha. \tag{1}$$

• The stress-energy tensor of the electromagnetic field is

$$T_{\rm em}^{\alpha\beta} = F^{\alpha\delta} F^{\beta}{}_{\delta} - \frac{1}{4} g^{\alpha\beta} F^{\delta\gamma} F_{\delta\gamma}. \tag{2}$$

• The components of the Riemann tensor in a coordinate basis take the following expression in terms of the Christoffel symbols:

$$R^{\mu}{}_{\nu\lambda\sigma} = \Gamma^{\mu}{}_{\nu\sigma,\lambda} - \Gamma^{\mu}{}_{\nu\lambda,\sigma} + \Gamma^{\mu}{}_{\rho\lambda}\Gamma^{\rho}{}_{\nu\sigma} - \Gamma^{\mu}{}_{\rho\sigma}\Gamma^{\rho}{}_{\nu\lambda}. \tag{3}$$

• The Ricci tensor is given by the following contraction of the Riemann tensor:

$$R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta}. \tag{4}$$

The two problems below are not easy, but you will be guided through them step-by-step, and hopefully, you'll find them interesting. Good luck!

PROBLEM 1: GEOMETRIC OPTICS IN CURVED SPACETIME

In a previous homework, you have derived that in the Lorenz gauge $\nabla^\alpha A_\alpha = 0$, the vector potential satisfies the following linear differential equation in the absence of sources (i.e. for $J_\alpha = 0$):

$$(\mathcal{L}A)_\alpha \equiv \nabla^\beta \nabla_\beta A_\alpha - R^\gamma{}_\alpha A_\gamma = 0. \tag{5}$$

Pick some coordinate system, and write A_μ as the real part of a complex vector $\mathcal{A}_\mu$ with complex amplitude a_μ and phase θ:

$$A_\mu = \text{Re}[\mathcal{A}_\mu], \quad \mathcal{A}_\mu \equiv a_\mu \text{e}^{i\theta}. \tag{6}$$

Since the equation satisfied by A_α is real and linear, we may first find complex solutions of $(\mathcal{L}\mathcal{A})_\mu = 0$ (with the Lorenz gauge condition $\nabla^\mu \mathcal{A}_\mu = 0$) and then take the real part. We also define the (real) wavevector k_μ as the gradient of the phase:

$$k_\mu \equiv \nabla_\mu \theta. \tag{7}$$

The wavevector k^μ is tangent to the light rays, which are the curves normal to the surfaces of constant phase.

Let us denote by $\mathcal{R} \sim (\text{Riemann})^{-1/2}$ the characteristic radius of curvature of spacetime. We assume that the amplitude a_μ and wavevector k_μ vary on a characteristic scale comparable to $\mathcal{R}$, but that the phase θ varies on a much shorter scale $\lambda \ll \mathcal{R}$:

$$\Big|\frac{\nabla_\mu a_\nu}{a_\nu}\Big| \sim \frac{1}{\mathcal{R}} \sim \Big|\frac{\nabla_\mu k_\nu}{k_\nu}\Big|, \qquad |k_\mu| \sim \frac{1}{\lambda} \gg \frac{1}{\mathcal{R}}, \tag{8}$$

where the equations apply to individual components (not to the norms of vectors).

(i) Show that, to leading order in $\mathcal{R}/\lambda$, the Lorenz gauge condition $\nabla^\mu \mathcal{A}_\mu = 0$ implies that the wavevector is orthogonal to the complex amplitude, i.e.

$$a^\mu k_\mu = 0. \tag{9}$$

(ii) Write out explicitly the propagation equation $\nabla^\nu \nabla_\nu \mathcal{A}_\mu - R^\nu{}_\mu \mathcal{A}_\nu = 0$ in terms of the complex amplitude and wavevector. Show that to leading order in $\mathcal{R}/\lambda$, this equation implies that the wavevector is null, i.e.

$$k^\mu k_\mu = 0, \tag{10}$$

and that at next-to-leading order, it implies

$$k^\nu \nabla_\nu a_\mu = -\frac{1}{2}(\nabla_\nu k^\nu) a_\mu. \tag{11}$$

(iii) Using $k^\mu k_\mu = 0$ and the fact that the wavevector is the gradient of a scalar, show that *light rays are null geodesics*, i.e. that their tangent vector k^μ is parallel-transported along the rays.

(iv) For simplicity we will assume that $a_\mu \overline{a}^\mu > 0$, where $\overline{a}^\mu$ is the complex congugate of a^μ. We define the scalar amplitude $a \equiv (a_\mu \overline{a}^\mu)^{1/2}$, and the polarization vector $e^\mu \equiv a^\mu / a$, which satisfies $e_\mu \overline{e}^\mu = 1$ and $k^\mu e_\mu = 0$. Using Eq. (11), first derive the equation satisfied by the scalar amplitude a, then show that *the polarization vector is parallel-transported along light rays.*

(v) Again using Eq. (11) and the equation you found for a, show that $a^2 k^\mu$ is a conserved current. This is the law of *conservation of photon number*, as we will understand in the last question.

(vi) Given $X = \text{Re}\left[x\text{e}^{i\theta}\right]$ and $Y = \text{Re}\left[y\text{e}^{i\theta}\right]$, where x and y are slowly-varying complex amplitudes, and θ is a rapidly-varying phase, it is easy to show that the average of XY over several wavelengths of the phase is $\langle XY \rangle = \frac{1}{2} x\overline{y}$. Using this result, and those derived so far, show that the electromagnetic stress-energy tensor, averaged over several wavelengths, and to leading order in $\mathcal{R}/\lambda$, is given by

$$\langle T^{\mu\nu}_{\text{em}} \rangle = \frac{1}{2} a^2 k^\mu k^\nu. \tag{12}$$

Using the results derived earlier, show that this is indeed conserved, i.e. divergence-less. Assuming a photon has 4-momentum $p^\mu = \hbar k^\mu$, explain why $\nabla_\mu(a^2 k^\mu) = 0$ can be interpreted as photon number conservation, and explicitly write an expression for the photon-number current (use the physical meaning of the stress-energy tensor).

PROBLEM 2: PERTURBATION ABOUT A CURVED BACKGROUND

In a specific coordinate system (which we'll refer to as "the original coordinate system"), suppose the coefficients of the metric take the form

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu}, \tag{13}$$

where the characteristic values of $h_{\mu\nu}$ are much smaller than those of $\overline{g}_{\mu\nu}$: $|h_{\mu\nu}| \ll |\overline{g}_{\mu\nu}|$. We assume that $\overline{g}_{\mu\nu}$ is non-degenerate. The bar of $\overline{g}_{\mu\nu}$ **does not** mean trace-reversed, it is our notation for the "background metric". The goal of this problem is to show that the components of the Ricci tensor, to linear order in perturbations, take the form

$$R_{\mu\nu} = \overline{R}_{\mu\nu} + h^{\sigma}{}_{(\mu|\nu)\sigma} - \frac{1}{2}\left(h_{|\mu\nu} + h_{\mu\nu|\sigma}{}^{\sigma}\right), \tag{14}$$

where $\overline{R}_{\mu\nu}$ is the Ricci tensor computed from the background metric, and $|$ denotes a covariant differentiation with respect to the background metric. In this expression, indices of $h_{\mu\nu}$ are raised with the inverse background metric $\overline{g}^{\mu\nu}$, and $h \equiv h^{\sigma}{}_{\sigma}$ is the trace of $h_{\mu\nu}$ (so the first term inside the parenthesis is $h^{\sigma}{}_{\sigma|\mu\nu}$). We will *not* use the fact that $h_{\mu\nu}$ is a perturbation until the very last question of the problem.

(i) Although this background + perturbation split is only valid in a specific coordinate system, explain how one can define a tensor field $\overline{g}_{\alpha\beta}$ whose components in this coordinate system are $\overline{g}_{\mu\nu}$. One can think of $\overline{g}_{\alpha\beta}$ as another metric coexisting with $g_{\alpha\beta}$. Since the metric $g_{\alpha\beta}$ is itself a tensor field, then $h_{\alpha\beta} \equiv g_{\alpha\beta} - \overline{g}_{\alpha\beta}$ is also a tensor field, whose components in the original coordinate system are $h_{\mu\nu}$.

(ii) We denote by ∇ the usual covariant derivative (i.e. the torsion-free covariant derivative compatible with $g_{\alpha\beta}$) and by $\overline{\nabla}$ the (torsion-free) covariant derivative compatible with $\overline{g}_{\alpha\beta}$, i.e. such that $\overline{\nabla}_{\gamma}\overline{g}_{\alpha\beta} = 0$. In an arbitrary coordinate system, what are the components of $\overline{\nabla}_{\alpha}V^{\beta}$, where V^{β} is a vector field? Give the explicit expression of the connection coefficients $\overline{\Gamma}^{\lambda}{}_{\mu\nu}$.

(iii) Explain why and in which sense the difference $\boldsymbol{S} \equiv \nabla - \overline{\nabla}$ is a tensor of rank $(1,2)$. What are its components in an arbitrary coordinate system, in terms of the derivatives of the components of the two metrics?

(iv) Compute the components of $S^{\alpha}{}_{\beta\gamma}$ in a locally-inertial coordinate system *of the background metric*, and show that they can be rewritten as

$$S^{\lambda}{}_{\mu\nu} = g^{\lambda\sigma}\left(h_{\sigma(\mu|\nu)} - \frac{1}{2}h_{\mu\nu|\sigma}\right). \tag{15}$$

Explain why this expression holds, in fact, in any coordinate system.

(v) Define $\overline{R}^{\alpha}{}_{\beta\gamma\delta}$ to be the Riemann tensor built from $\overline{g}_{\alpha\beta}$ and $\overline{\nabla}$, and $R^{\alpha}{}_{\beta\gamma\delta}$ the Riemann tensor associated with the true metric $g_{\alpha\beta}$. Compute the components of these two tensors in a locally-inertial coordinate system *of the background metric*, and show that, in this coordinate system, and in fact, in any coordinate system,

$$R^{\alpha}{}_{\beta\gamma\delta} - \overline{R}^{\alpha}{}_{\beta\gamma\delta} = S^{\alpha}{}_{\beta\delta|\gamma} - S^{\alpha}{}_{\beta\gamma|\delta} + S^{\alpha}{}_{\rho\gamma}S^{\rho}{}_{\beta\delta} - S^{\alpha}{}_{\rho\delta}S^{\rho}{}_{\beta\gamma}. \tag{16}$$

(vi) Contract to get the Ricci tensor. Finally, express the components of the Ricci tensor in the original coordinate system, and expand to linear order in $h_{\mu\nu}$ to derive Eq. (14). You will need to re-express $g^{\lambda\sigma}$ in terms of $\overline{g}^{\lambda\sigma}$ and use the consequence of metric compatibility for the covariant derivative of the inverse metric.

Later on we will use this result to derive the propagation of gravitational waves on a curved background spacetime, in a similar way as we did for electromagnetic waves in Problem 1.