# General Relativity Fall 2018
# Homework 1

due September 11th 2018

**Exercise 1: Uniformly accelerated motion**

Consider a particle with 4-velocity $u^\mu = dx^\mu/d\tau$ and 4-acceleration $a^\mu = du^\mu/d\tau$, constant in the particle's rest frame.

($i$) Compute $u^\alpha a_\alpha$. Use this simple result to write the generic form of the 4-acceleration in the particle's rest-frame.

($ii$) In an inertial frame of reference, in which the particle is initially at rest at the origin, describe the particle's motion, i.e. compute its 3-velocity, position, and proper time as a function of coordinate time $t$. Discuss how these results match with the non-relativistic limit.

**Exercise 2: Thomas precession**

($i$) Consider a massive particle with 4-velocity $u^\alpha$. If $S^\alpha$ is a 4-vector defined along the particle's worldline, explain in words what the condition $u_\alpha S^\alpha = 0$ implies [I am looking for a sentence that contains "in the particle's rest frame"]. How about the conditions $(\eta_{\alpha\beta} + u_\alpha u_\beta)S^\beta = 0$?

($ii$) The spin of a particle can be represented by a spacelike 4-vector $S^\alpha$ orthogonal to $u_\alpha$ and with constant norm $(S^\alpha S_\alpha)^{1/2}$. Assuming there are no torques acting on the particle, show that the rate of change of $S^\alpha$ along the particle's worldline is given by $dS^\alpha/d\tau = (S^\beta a_\beta)u^\alpha$ [*hint*: argue that this Lorentz-vector can only be constructed out of contractions of $u^\alpha$ and $a^\alpha$; then show that these contractions must take the form $dS^\alpha/d\tau = Au^\alpha + Ba^\alpha$; find $A$ and $B$ from the requirement that $S^\alpha$ remains orthogonal to $u^\alpha$ and keeps a constant norm.]

**Exercise 3: Index manipulation**

($i$) If the tensor $T_{\alpha\beta}$ is symmetric, show that $T^\alpha{}_\beta = T_\beta{}^\alpha$.

($ii$) Given a rank (0,2) tensor $T_{\alpha\beta}$, what is the rank of the tensor $T_{\alpha\beta}T_\gamma{}^\sigma T^{\beta\gamma}$ ? How about $T_{\alpha\beta}T_\gamma{}^\alpha T^{\beta\gamma}$ ?

($iii$) Suppose that in some inertial frame, the components of the tensor $T_{\alpha\beta}$ are given by

$$T_{\mu\nu} = \begin{pmatrix} 2 & 0 & -1 & 1 \\ 3 & 0 & 2 & 1 \\ -2 & 1 & 0 & -1 \\ -1 & 0 & -2 & 3 \end{pmatrix}. \tag{1}$$

Compute $T_{\alpha(\beta}T_{\gamma)}{}^\alpha T^{[\beta\gamma]}$.

**Exercise 4: Ideal fluid**

($i$) Recall that the components of the 4-velocity in a generic inertial frame are given by $u^\mu = \gamma(1, \vec{v})$, $\gamma \equiv (1-v^2)^{-1/2}$. We will see next week that the stress-energy tensor of an ideal fluid with rest-frame density $\rho$, pressure $P$ and 4-velocity $u^\mu$ is given by

$$T^{\mu\nu} = \rho u^\mu u^\nu + P(\eta^{\mu\nu} + u^\mu u^\nu). \tag{2}$$

Starting from the conservation of the stress-energy tensor $\partial_\mu T^{\mu\nu} = 0$, write explictly the relativistic ideal fluid equations in terms of the fluid's rest-frame density $\rho$, pressure $P$ and 3-velocity $\vec{v}$.

($ii$) We assume that the ideal fluid is made of identical particles whose number is conserved. We define the particle number-current 4-vector $N^\alpha = nu^\alpha$, where $n$ is the fluid's proper number density and $u^\alpha$ the fluid's 4-velocity. Combining the conservation of particle number $\partial_\alpha N^\alpha = 0$ with the conservation of energy-momentum, show that the specific entropy $\sigma$, defined through the differential relation $Td\sigma \equiv Pd(1/n) + d(\rho/n)$, is constant along the fluid's motion. [*Hint*: a quantity $X$ is constant along the fluid's motion if $\partial_t X + \vec{v} \cdot \partial_{\vec{x}} X = 0$].