General Relativity Fall 2018
Homework 2

due September 18th 2017

Exercise 1: gravitational redshift

What is the gravitational redshift of light emitted from the surface of the Sun? From that of a white dwarf? And a neutron star? I only need an order-of-magnitude estimate for this problem (your answer should not have more than one significant digit).

Exercise 2: gravitomagnetism

There is an obvious analogy between electrostatics and Newtonian gravitational fields:

$$\vec{\nabla} \cdot \vec{E} = \rho_e, \qquad m\frac{d\vec{v}}{dt} = q\vec{E}, \tag{1}$$

$$\vec{\nabla} \cdot \vec{g} = -4\pi G\rho_{\rm m}, \qquad m\frac{d\vec{v}}{dt} = m\vec{g}, \tag{2}$$

where ρ_e and ρ_m are the density of electric charge and rest-mass, respectively, and $\vec{g}$ is the Newtonian gravitational field. Suppose gravity was just like electromagnetism, and that we simply never realized that there exists a gravito-magnetic force, analogous to the magnetic force. (i) Complete the equations above to include such a gravito-magnetic aspect (note that, in a few lectures, we will actually derive such equations from GR in the weak-field limit). (ii) Calculate the time-independent gravito-magnetic field (let us call it $\vec{b}$) generated by the Sun, as a function of distance, assuming for simplicity that the sun is in solid rotation. This should be similar to the magnetic field generated by a magnetic dipole moment.

(iii) Consider a circular Keplerian orbit around the Sun. The angular momentum per unit mass is $\vec{\ell} = \vec{r} \times \vec{v}$. Calculate at what rate the angular momentum precesses due to the gravito-magnetic force, over many orbital periods. *Hint*: We want $\langle d\vec{\ell}/dt \rangle$, where $\langle ... \rangle$ is the average over an orbital period. Before taking the average, express $\vec{v}$ in terms of $\vec{\ell}$ and $\vec{r}$ for a circular orbit. You will then have to compute $\langle r_i r_j \rangle$. This can only depend on δ_{ij}, ℓ_i. Your final result should look like $\langle d\vec{\ell}/dt \rangle = \vec{\omega} \times \vec{\ell}$.

(iv) Estimate the precession timescale at the orbit of Mercury. Can this effect explain the precession of Mercury's perihelion?

Exercise 3: orthonormal bases

Conside a vector space of dimension n and a metric $g_{\mu\nu}$ on that space. Here we consider the more general notion of a metric (*not* the Minkowski metric), i.e. $g_{\mu\nu}$ is a symmetric tensor of rank $(0, 2)$ that is non-degenerate, i.e. such that for any vector X^μ, $g_{\mu\nu}X^\nu = 0 \Rightarrow X^\mu = 0$. Prove that there exist coordinates $Y^{\mu'} = \Lambda^{\mu'}{}_{\mu} X^\mu$ such that

$$g_{\mu\nu}X^{\mu\nu} = \sum_{\mu'} \pm (Y^{\mu'})^2, \tag{3}$$

i.e. a sum of (plus or minus) squares.

Hint: work with the quadratic form $Q(X^1, ..., X^n) \equiv g_{\mu\nu}X^\mu X^\nu$ and proceed recursively. First consider the case where one of the diagonal components does not vanish, say g_{11}, and find a new coordinate $\tilde{X}^1$ such that $Q(X^1, ..., X^n) = \pm(\tilde{X}^1)^2 + \tilde{Q}(X^2, ..., X^n)$. Second, consider the case where all the diagonal components vanish, and suppose $g_{12} \neq 0$. Find new coordinates $\tilde{X}^1, \tilde{X}^2$ such that $Q(X^1, ..., X^n) = (\tilde{X}^1)^2 - (\tilde{X}^2)^2 + \tilde{Q}(X^3, ..., X^n)$.