General Relativity Fall 2018
Homework 3

due Thursday, September 27th 2018

Exercise 1: transformation of connection coefficients

Let us pick a generic covariant derivative ∇ (it need not be *the* torsion-free, metric compatible covariant derivative for this argument). Given some coordinates $\{x^\mu\}$, the covariant derivative of a vector V has components

$$\nabla_\nu V^\mu \equiv (\nabla V)^\mu{}_\nu = \frac{\partial V^\mu}{\partial x^\nu} + \Gamma^\mu{}_{\nu\lambda} V^\lambda, \tag{1}$$

where $\Gamma^\mu{}_{\nu\lambda}$ is the *connection* between the covariant derivative and the partial derivative *in that coordinate system.* Under a change of coordinates, the components of vectors and tensors transform in the usual way with $\Lambda^{\mu'}{}_\mu \equiv \partial x^{\mu'}/\partial x^\mu$ (note that the matrix $\Lambda^{\mu'}{}_\mu$ is *not* a Lorentz transformation and depends on position).

(i) Given that ∇V is a tensor of rank (1, 1) and that V is a vector, derive the transformation law for the coefficient $\Gamma^\mu{}_{\mu\lambda}$ under a change of coordinates.

(ii) Prove that the antisymmetric part of the connection coefficient, $\Gamma^\mu{}_{[\nu\lambda]}$ is a tensor (even though the connection itself is not).

Exercise 2: metric-compatible covariant derivative

(i) Using Equation (1) above, derive the expression for the components of the covariant derivative of a tensor of rank (0, 2), $\nabla_\lambda T_{\mu\nu} \equiv (\nabla T)_{\mu\nu\lambda}$. You only need to use the following properties of a covariant derivative: 1) if applied to a scalar field f, its components are just the regular partial derivative, $\nabla_\mu f = \partial_\mu f$, and 2), it satisfies Leibniz' rule.

(ii) Applying this to the metric tensor, derive an expression for the symmetric part $\Gamma^\mu{}_{(\lambda\nu)}$ of the connection coefficient of a metric-compatible covariant derivative (i.e. a covariant derivative that vanishes when applied to the metric tensor), as a function of the metric and of the torsion tensor $T^\mu{}_{\lambda\nu} \equiv 2\Gamma^\mu{}_{[\lambda\nu]}$. Use the symmetric/antisymmetric notation to make your final expression as compact as possible. *hint*: Your expression should reduce to the Christoffel symbol for a torsion-free connection. Be careful with the ordering of indices!

Exercise 3: Christoffel symbols of the FLRW metric

The Friedmann-Lemaitre-Robertson-Walker (FLRW) metric describes a homogeneous and isotropic expanding Universe:

$$ds^2 = a(t)^2 \left(-dt^2 + dx^2 + dy^2 + dz^2\right). \tag{2}$$

Compute the Christoffel symbols for this metric. Instead of computing all 24 coefficients one by one, use the simplicity of the metric to derive general expressions first. Your grade on this question requires not only getting the right answer, but also having a compact derivation.