# General Relativity Fall 2018
# Homework 4

due Thursday, October 4th 2018

**Exercise1: Levi-Civita tensor**

We define the Levi-Civita tensor $\epsilon_{\alpha\beta\gamma\delta}$ as the completely antisymmetric tensor whose 0123 component is 1 in a locally inertial coordinate system. Compute its components in an arbitrary coordinate system.

**Exercise 2: minimal substitution subtleties**

In curved spacetime Maxwell's equations are

$$\nabla^\alpha F_{\alpha\beta} = -J_\beta, \tag{1}$$

$$\nabla_{[\alpha} F_{\beta\delta]} = 0. \tag{2}$$

The second equation implies that one can find a vector potential $A^\alpha$ such that $F_{\alpha\beta} = 2\nabla_{[\alpha} A_{\beta]}$.

What is the equation satisfied by the vector potential in the Lorenz gauge $\nabla_\alpha A^\alpha = 0$?

**Exercise 3: Curvature of a sphere**

The 2-sphere of radius $r$ has metric

$$ds^2 = r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{3}$$

$(i)$ Compute the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ by using their explicit definition in terms of the metric and its derivatives.

$(ii)$ Derive the geodesic equations and read off the Christoffel symbols by extremizing $\int d\tau_* g_{\mu\nu}(dx^\mu/d\tau_*)(dx^\nu/d\tau_*)$. You should get the same answer as in $(i)$.

$(iii)$ Compute the Ricci scalar of this spacetime.

$(iv)$ Write the equations of parallel transport of a tangent vector $V = V^\theta\partial_\theta + V^\varphi\partial_\varphi$ along a small circle with constant $\theta = \theta_0$.

$(v)$ Solve these equations, starting with initial condtions $V_0^\theta$ and $V_0^\varphi$ at $\varphi = 0$. How does the vector compare to itself after being parallel-transported once around a circle?

**Exercise 4: Rindler spacetime**

Consider a spacetime with two-dimensional metric

$$ds^2 = -x^2 dt^2 + dx^2, \tag{4}$$

where the coordinates $(t, x)$ are defined over $-\infty < t < \infty$ and $0 < x < \infty$.

$(i)$ Compute the connection coefficients of this metric with your favorite method.

$(ii)$ Compute the Ricci scalar of this metric. What does your result imply? Make sure to invoque the Riemann tensor in your argumentation.

$(iii)$ What is the acceleration 2-vector of observers of constant $x$?

$(iv)$ Show that null geodesics are such that $t = \pm\ln(x)+$ constant.

$(v)$ Define the coordinates $u \equiv t - \ln(x)$ and $v \equiv t + \ln(x)$. What are the null geodesics in these coordinates? What are the metric coefficients in these coordinates?

$(vi)$ Can you find new coordinates $T, X$ in which $ds^2 = -dT^2 + dX^2$? [*Hint*: start by finding coordinates $U, V$ in which $ds^2 = -dUdV$]. Give an explicit expression of the old coordinates in terms of the new coordinates.

$(vii)$ In a spacetime diagram in the $(T, X)$ coordinates, draw the worldlines of observers of constant $x$, and those of observers of constant $t$.