General Relativity Fall 2018
Homework 5

due Thursday, October 11th 2018

Exercise 1: $f(R)$ **gravity**

Derive the field equations in the $f(R)$ theory of gravity, for which the action is given by

$$S = \frac{1}{16\pi} \int d^4x \sqrt{|g|}\ f(R)\ + S_M, \tag{1}$$

where an arbitrary function of the Ricci scalar. You should arrive at an equation of the form $\tilde{G}_{\mu\nu} = 8\pi T_{\mu\nu}$, where $\tilde{G}_{\mu\nu}$ reduces to the Einstein field equation when $f = R$. Check explicitly that $\nabla_\mu \tilde{G}^{\mu\nu} = 0$, as is required for $T_{\mu\nu}$ to be conserved.

Exercise 2: constraint equations

Using the contracted Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$, show that the 0μ (both indices up) Einstein field equations do not contain any second time derivatives: these are pure constraint equations. [Hint: rewrite $\nabla_\mu G^{\mu\nu}$ in the form $\partial_0 G^{0\nu} = S^\nu$, and argue that S^ν contains at most two time derivatives].

Exercise 3: Killing vector fields and conservation laws

(i) Prove that if the metric components do not depend on a specific coordinate $x^{\sigma *}$, i.e. $\forall \mu, \nu,\ \partial_{\sigma *} g_{\mu\nu} = 0$, then the component $p_{\sigma *}$ of the (covariant) 4-momentum is conserved along geodesics.

(ii) Prove that if a vector field K^μ satisfies **Killings' equation**, $\nabla_{(\mu} K_{\nu)} = 0$, then $K_\nu p^\nu$ is conserved along geodesics with 4-momentum p^μ. Such a field is called a Killing vector field.

(iii) Prove that, given a conserved stress-energy tensor $T_{\mu\nu}$, and a Killing vector field K^μ, the current $J^\mu \equiv K_\nu T^{\mu\nu}$ is conserved (i.e. divergence-free).

(iv) Show that a Killing vector field K^ρ satisfies $\nabla_\mu \nabla_\sigma K^\rho = R^\rho{}_{\sigma\mu\nu} K^\nu$, and as a consequence, that the Ricci scalar is constant along curves with tangent K^μ, i.e. $K^\mu \nabla_\mu R = 0$. This conveys the notion that the geometry is not changing along a Killing vector field.

(v) Prove that if $K = \partial_{(\sigma *)}$, then $\partial_{\sigma *} g_{\mu\nu} = 2\nabla_{(\mu} K_{\nu)}$. This implies that the metric is independent of a coordinate $x^{\sigma *}$ if and only if the corresponding vector field $K = \partial_{(\sigma *)}$ is a Killing vector field. *Hint: You will need to use the fact that* $g_{\mu\nu} = g_{\alpha\beta} \partial^\alpha_{(\mu)} \partial^\beta_{(\nu)}$ *and that the commutator of oordinate basis vectors vanishes.*