General Relativity Fall 2018
Homework 6

due October 18th 2017

Exercise 1: linearized Einstein tensor

In class we derived the form of the linearized Einstein tensor, up to an overall normalization constant:

$$G_{\mu\nu} = A\left(\Box h_{\mu\nu} - 2\, \partial^\alpha \partial_{(\mu} h_{\nu)\alpha} + \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h\right). \tag{1}$$

We can compute the coefficient A explicitly by picking as simple a metric as possible, for instance

$$ds^2 = (-1 + h_{00})dt^2 + \delta_{ij}dx^i dx^j. \tag{2}$$

(i) Explain why the above metric would have a vanishing Einstein tensor if h_{00} only depends on t.

(ii) Compute the Ricci scalar for the above metric, if $h_{00}(x)$ is a function of the first spatial coordinate only, and find the value of A.

Exercise 2: Harmonic gauge

Given a metric perturbation $h_{\mu\nu} \ll 1$, we denote by $\overline{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ the trace-reversed perturbation, where $h \equiv \eta^{\mu\nu}h_{\mu\nu}$.

(i) For a scalar function f, express $\Box f \equiv \nabla^\mu\nabla_\mu f$ in terms of partial derivatives of f and $\overline{h}_{\mu\nu}$. You should get a simple expression with only two terms.

(ii) Substitute $f = x^{\sigma *}$, a specific coordinate, in the above expression, and show that $\Box x^{\sigma *} = 0$ in the harmonic gauge defined as $\partial^\mu \overline{h}_{\mu\nu} = 0$. The coordinates are therefore harmonic functions in this gauge, hence its name.

(iii) Given the metric perturbation $h_{\mu'\nu'}$ in some initial primed coordinate system $\{x^{\mu'}\}$, derive explicitly the gauge transformation $x^\mu = x^{\mu'} - \xi^\mu$ required to enforce the harmonic gauge conditions $\partial^\mu \overline{h}_{\mu\nu} = 0$.

(iv) Simplify the Einstein tensor in the harmonic gauge.

Exercise 3: Geometric units

Geometric units are units in which $G = c = 1$. In those units, masses, lengths and times have the same dimensions.

(i) Explicitly using G and c, how does one convert a mass M to a length? To a time?

(ii) Convert the masses of the Earth and Sun to centimeters.

(iii) In geometric units, what is the Newtonian gravitational potential Φ at the surface of the Earth, in the simplest unit? At the surface of the Sun?

(iv) We saw that $R_{0i0j} \approx \partial_i\partial_j\Phi$ in the Newtonian limit, so that the Ricci scalar is $R \sim \nabla^2\phi \sim 4\pi\rho$. We also saw that the Ricci scalar is order $1/\mathcal{R}^2$, where $\mathcal{R}$ is the radius of curvature (see Homework 4). What is the characteristic radius of curvature of spacetime near Earth? And near the Sun?

(v) The reduced Planck constant $\hbar \approx 10^{-27}$ g cm^2 s^{-1}. In geometric units, what is the value of $\hbar^{1/2}$ in grams? In centimeters? In seconds? These are the Planck mass, length, and time, respectively.

Exercise 4: black hole perils

Estimate how close one can get from a point mass M before tidal forces on the human body are sufficiently strong to become lethal. For what mass M does this distance $L_{\rm lethal}(M)$ become smaller than M (in geometric units)?