General Relativity Fall 2018
Homework 7

due October 25th 2018

Exercise 1: No gravitational waves in 2 + 1 dimension

Explain why there are no gravitational waves in general relativity in 2 + 1 dimensions (i.e. one time dimension and 2 spatial dimensions).

Exercise 2: Precession of gyroscopes in a weak gravitational field

Consider the following weak-field metric outside a source with mass M and angular momentum $\vec{J}$:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{4}{r^2}(\hat{x} \times \vec{J}) \cdot d\vec{x} dt + \left(1 + \frac{2M}{r}\right) d\vec{x}^2 \equiv (\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu. \tag{1}$$

Consider a massive and non-relativistic point particle on a geodesic, i.e. on a nearly Keplerian orbit about the central mass. Suppose it carries a spin S^α that is a purely spatial vector in the rest-frame of the particle, i.e. $S^\alpha U_\alpha = 0$. This spin is parallel-transported along the particle's geodesic, i.e. $U^\beta \nabla_\beta S^\alpha = 0$.

(i) Compute S^0 as a function of S^i, to lowest order in the particle's velocity $\vec{v} \equiv d\vec{x}/dt$ and metric perturbations.

(ii) Write explicitly the parallel-transport equation for S^i in terms of coordinate time t, and to linear order in the particle's velocity. For now keep the expression general in terms of Christoffel symbols.

(iii) Compute the relevant Christoffel symbols to lowest order in metric perturbations, and simplify the parallel-transport equation for S^i. Express your final result in terms of $x^{(i}v^{j)}$ and $x^{[i}v^{j]}$.

(iv) Re-express the antisymmetric pieces in terms of the orbital specific angular momentum $\vec{\ell} = \vec{x} \times \vec{v}$. Recall that $\frac{d\vec{v}}{dt} = -\frac{M}{r^2}\hat{x}$ for a Keplerian orbit; use this to express the symmetric pieces $x^{(i}v^{j)}$ in terms of a time derivative.

(v) Average over one circular orbit, assuming S^i changes little in one orbit (which you will confirm a posteriori). Use the fact that the orbit is periodic to argue that the total derivative term cancels. You should get something of the form

$$\left\langle \frac{d\vec{S}}{dt} \right\rangle = \vec{\Omega} \times \vec{S}, \tag{2}$$

where $\vec{\Omega}$ contains two terms: one linear in $M\vec{\ell}$ and the other linear in $\vec{J}$. The former is the **geodetic precession** and the latter is **Lense-Thirring precession**, applied this time to a gyroscope rather than to the orbital angular momentum as we did in class.

(vi) Estimate the two precession rates for a point mass orbiting the Earth at a 650-km altitude on an circular polar orbit (as is the case for the Gravity Probe B (GPB) satellite), and express them in arcsecond per year.

For reference, GPB has measured the geodetic precession to a fractional accuracy of 0.003 and the Lense-Thirring gyroscopic precession to a fractional accuracy of 0.2. The LAGEOS satellites have measured the orbital Lense-Thirring precession to a fractional accuracy of 0.05.

Exercise 3: Perturbed $f(R)$ gravity

(i) In the harmonic gauge, derive the linearized Einstein field equations in $f(R)$ gravity (starting from the result of HW 5), assuming $f(R) = R + \frac{1}{2}L^2R^2$, where L is a characteristic lengthscale.

(ii) Solve this equation *in Fourier space* for the metric perturbation generated by a stationary source, whose only non-vanishing component of the stress-energy tensor is $T_{00} = \rho$. Show that it amounts to having a scale-dependent Newton constant in the Poisson equation.