General Relativity Fall 2018
Homework 8

due November 1st 2018

Exercise 1: Detectability of gravitational waves

Consider an equal-mass binary on a circular orbit, with total mass $M = mM_\odot$ (i.e. m is the mass in solar mass units). Suppose the binary is at a distance $d = 100\ d_{100}$ Mpc away (i.e. d_{100} is the distance in units of 100 Mpc).

(i) Estimate the characteristic frequency and strain h_{ij} of gravitational waves when the semimajor axis is $a \approx 10M$. Give a numerical result, expressed in terms of m and d_{100}.

(ii) Suppose a GW detector has a sensitivity to strains h_{ij} as low as 10^{-21} for frequencies in the range $10^2 - 10^3$ Hz (this is, very roughly, the case for LIGO). Using your estimates above (i.e. for $a \approx 10\ M$), estimate the characteristic masses to which the detector is sensitive. As a function of mass, what is the maximum distance at which a binary can be detected?

Exercise 2: Order-of-magnitude estimates

(i) Suppose you wave your arms wave your arms at a frequency of 1 Hz. Estimate the wavelength of gravitational waves that you radiate and the characteristic amplitude of the GW strain one wavelength away.

(ii) We will derive next week that gravitational waves carry energy, and that the power radiated in GWs by a time-varying quadrupole moment is $P \sim \dddot{Q}_{ij}\dddot{Q}_{ij}$. Suppose that the gravitational-wave energy is carried by individual gravitons of energy $h_\mathrm{P}\nu$, where h_P is Planck's constant and ν is the frequency. When moving your arms as in (i), how long does it take to radiate one single graviton?

(iii) Suppose a *perfectly spherically symmetric* star of mass M and initial radius R collapses to a point in a free-fall time. What is the characteristic strain of GWs that this process produces at large distances?

Exercise 3: frequency shifts by gravitational waves

Consider the metric perturbation due to a single Fourier mode of a GW:

$$ds^2 = -dt^2 + (\delta_{ij} + h_{ij}^{\mathrm{TT}})dx^i dx^j, \tag{1}$$

$$h_{ij}^{\mathrm{TT}}(t, \vec{x}) = A_{ij}\cos(\omega t - \vec{k}\cdot\vec{x}), \quad |\vec{k}| = \omega, \quad A_{ij}k^j = 0. \tag{2}$$

(i) Compute the Chirstoffel symbols of this metric, to linear order in h_{ij}.
(ii) Consider a massless particle on a geodesic, emitted at $(t_\mathrm{em}, \vec{x}_\mathrm{em})$. Solve the geodesic equation to linear order in h_{ij}, and compute p^0 at some later event $(t_\mathrm{obs}, \vec{x}_\mathrm{obs})$ on the geodesic.