General Relativity Fall 2018
Homework 9

due November 8th 2018

Cross section for capture through gravitational-wave radiation

In Newtonian theory two isolated point masses that are initially unbound would always remain so, as the total energy of the isolated system is conserved. General relativity tells us that it is possible to radiate away energy in the form of gravitational waves. If the radiated energy is large enough, it is possible for two initially unbound point masses to become bound. The goal of this exercise is to estimate how the cross section for this process scales with the masses and relative velocity at infinity.

Consider two point masses M_1, M_2 in a hyperbolic (hence unbound) orbit, with impact parameter b and relative velocity at infinity v_∞. We denote by $M \equiv M_1 + M_2$ the total mass and by $m = M_1 M_2/M$ the reduced mass of the system. We place the origin of the coordinate system at the center of mass, and define the relative separation and velocity by $\vec{r} \equiv \vec{r}_2 - \vec{r}_1$ and $\vec{v} \equiv \dot{\vec{r}}$, respectively.

(*i*) Compute the inertia tensor $I_{ij} \equiv \int d^3y \; y_i y_j \; \rho$, and its first, second and third derivatives. Use the fact that $\dot{\vec{v}} = -(M/r^3)\vec{r}$ along the quasi-Newtonian trajectory to simplify your expression for $\dddot{I}_{ij}$ as much as possible – the final expression should only have two terms.

(*ii*) Compute $\dddot{Q}_{ij}\dddot{Q}_{ij}$, where $Q_{ij} \equiv I_{ij} - \frac{1}{3} I_{kk}\delta_{ij}$ is the quadrupole moment. I recommend first re-expressing $\dddot{Q}_{ij}\dddot{Q}_{ij}$ in terms of products of $\dddot{I}_{ij}$ instead of first computing $\dddot{Q}_{ij}$. Re-express your final expression in terms of the specific energy $\epsilon \equiv \frac{1}{2}v^2 - M/r$ and specific angular momentum $\ell \equiv |\vec{r} \times \vec{v}|$, which are constants of motion along the quasi-Newtonian orbit (these are the energy and angular momentum per unit reduced mass). Your final expression should only depend on m, M, ϵ, ℓ and r, and should have no more than 3 terms.

The power radiated in gravitational waves is $P = -\frac{1}{5}\dddot{Q}_{ij}\dddot{Q}_{ij}$. We could compute explicitly the total energy lost during the encounter, $\Delta E = \int dt P$. This would require using standard results for hyperbolic orbits and computing a few nasty integrals... Instead we will just get the overall scalings by approximating $\Delta E \sim P(r_p)\Delta t_p$, where r_p is the separation at pericenter (i.e. at closest approach) and Δt_p is the characteristic time spent near pericenter.

(*iii*) Using conservation of energy and angular momentum along the unperturbed orbit, express ϵ and ℓ in terms of v_∞ and r_p. Assuming that the orbit is quasi-parabolic, i.e. that $\epsilon \ll M/r_p$, estimate the power radiated near pericenter $P(r_p)$. You should have only one term, depending on m, M and r_p.

(*iv*) Using $\Delta t_p \sim r_p/v_p$, where v_p is the characteristic relative velocity at pericenter, estimate the total energy radiated in the encouter, $\Delta E \sim P(r_p)\Delta t_p$, and express this exclusively in terms of m, M and r_p (it will require estimating v_p).

(*v*) Using $\ell = b v_\infty$ and your result from (*iv*), re-express ΔE in terms of the impact parameter and velocity at infinity (and m, M). The cross section for capture is simply $\sigma = \pi b_{\max}^2$, where $b_{\max}$ is the maximum impact parameter for which $\Delta E < -m\epsilon$. Estimate $\sigma(v_\infty)$. You should get the following scaling

$$\sigma(v_\infty) \sim \left(\frac{m^2 M^{12}}{v_\infty^{18}}\right)^{1/7}. \tag{1}$$

(*vi*) Under what condition for the velocity is the quasi-parabolic assumption $\epsilon \ll M/r_p$ self-consistent?