General Relativity Fall 2018
Lecture 9: Einstein's field equation

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Fermi normal coordinates – Consider a fiducial timelike geodesic G. We will build a coordinate system (t, x^i) defined in a neighborhood of G, such that the geodesic has coordinates $x^i|_G = 0$, the metric is Minkowski along the geodesic, $g_{\mu\nu}|_G = \eta_{\mu\nu}$, and the first derivatives of the metric vanish along the geodesic, i.e. $\partial_\lambda g_{\mu\nu}|_G = 0$. This is more restrictive than a LICS: the metric is close to Minkowski not only at a point, but all along a curve!

Let us show that such a coordinate system exists. On the geodesic, we define $x^0 = \tau$, the proper time, which we initialize at zero at some point $p_0 \in G$. At that "initial" point, we put the metric in normal form, i.e. define three vector $e_{(1)}, e_{(2)}, e_{(3)}$, in addition to $e_{(0)} = d/d\tau = U$, the four-velocity at p_0, such that $g(e_{(\mu)}, e_{(\nu)})|_{p_0} = \eta_{\mu\nu}$. We then define the four 4-vectors $e_{(\mu)}$ all along G by parallel-transporting them (this is self-consistent for $e_{(0)} = U$, which is already parallel-transported along G). Since parallel transport preserves angles, we have $g(e_{(\mu)}, e_{(\nu)})|_G = \eta_{\mu\nu}$ all along the geodesic G.

Now, given four numbers $\{x^\mu\}$, we define the point $p(x^0, x^i)$ as follows. We build the unique geodesic starting on G at time $\tau = x^0$, with tangent vector $T = x^i e_{(i)}$. We parametrize the geodesic by λ, and define the point $p(x^\mu)$ to be the point reached when $\lambda = 1$. In other words, in some arbitrary coordinate system $\{y^\mu\}$, we solve for

$$\frac{d^2 y^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\sigma}\frac{dy^\nu}{d\lambda}\frac{dy^\sigma}{d\lambda} = 0, \tag{1}$$

$$y^\mu(\lambda = 0) = y^\mu(p(x^0, 0) \in G), \tag{2}$$

$$\frac{dy^\mu}{d\lambda}|_{\lambda=0} = x^i e^\mu_{(i)}. \tag{3}$$

This procedure does not necessarily work for arbitrary x^i: the geodesics we build do not necessarily extend to $\lambda = 1$!. However, for small enough x^i, the geodesics are indeed well defined up to $\lambda = 1$, which is a point close to G. For x^i small enough, we get $y^\mu \approx y^\mu(p(x^0, 0)) + x^i e^\mu_{(i)}$, which can be inverted to give $x^i \approx g_{\mu\nu}\Delta y^\mu e^\nu_{(i)}$. In other words, for any point close enough to G, there are unique coordinates x^μ.

What remains to show is that $\partial_{(\mu)}|_G = e_{(\mu)}$ and that the Chirstoffel symbols vanish on G, see the homework.

Since the curve G has coordinates $(x^0, 0, 0, 0)$, the Christoffel symbols are such that $\Gamma^\lambda_{\mu\nu}(x^0, 0) = 0$, implying $\partial_0\Gamma^\lambda_{\mu\nu}(x^0, 0) = 0$, i.e. the time-derivative of the Christoffel symbol vanishes along the curve G. Note that it is not the case for spatial derivatives.

Geodesic deviation – Last lecture we derived the formal equation for geodesic deviation. Let us rederive this using Fermi-normal coordinates $\{x^\mu\}$, centered on a fiducial geodesic G with coordinates $x^\mu_G = \{x^0_G, 0, 0, 0\}$. The spatial components of the geodesic equation for a particle in the vicinity of G is

$$\frac{d^2 x^k}{d\tau^2} = -\Gamma^k_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \approx -(x^\sigma - x^\sigma_G)\partial_\sigma\Gamma^k_{\mu\nu}|_G\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}. \tag{4}$$

The fiducial geodesic has $dx^\mu/d\tau = \delta^\mu_0$, so, to linear order, we get

$$\frac{d^2 x^k}{d\tau^2} \approx -(x^\sigma - x^\sigma_G)\partial_\sigma\Gamma^k_{00}. \tag{5}$$

Now recall that $\partial_0\Gamma^\lambda_{\mu\nu} = 0$ on the geodesic in Fermi normal coordinates, so we can subtract $\partial_0\Gamma^\lambda_{\sigma 0}$, and get

$$\frac{d^2 x^k}{d\tau^2} \approx -x^i R^k_{\ 0i0}. \tag{6}$$

So we see that the Riemann tensor plays the role of a **tidal field**.

Bianchi identity – The fully-covariant components of the Riemann tensor are

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda}\left(\partial_\mu\Gamma^\lambda_{\ \nu\sigma} - \partial_\nu\Gamma^\lambda_{\ \mu\sigma} + \Gamma^\lambda_{\mu\kappa}\Gamma^\kappa_{\nu\sigma} - \Gamma^\lambda_{\nu\kappa}\Gamma^\kappa_{\mu\sigma}\right). \tag{7}$$

We then have

$$\nabla_\gamma R_{\rho\sigma\mu\nu} = g_{\rho\lambda}\left(\partial_\gamma\partial_\mu\Gamma^\lambda{}_{\nu\sigma} - \partial_\gamma\partial_\nu\Gamma^\lambda{}_{\mu\sigma}\right) + \sim \Gamma\times R. \tag{8}$$

In a LICS, we therefore have

$$\begin{aligned}\nabla_\gamma R_{\rho\sigma\mu\nu} &= \eta_{\rho\lambda}\left(\partial_\gamma\partial_\mu\Gamma^\lambda{}_{\nu\sigma} - \partial_\gamma\partial_\nu\Gamma^\lambda{}_{\mu\sigma}\right) = \partial_\gamma\partial_\mu(\eta_{\rho\lambda}\Gamma^\lambda{}_{\nu\sigma}) - \partial_\gamma\partial_\nu(\eta_{\rho\lambda}\Gamma^\lambda{}_{\mu\sigma})\\ &= \frac{1}{2}\left(\partial_\gamma\partial_\mu\partial_\sigma g_{\rho\nu} + \partial_\gamma\partial_\mu\partial_\nu g_{\rho\sigma} - \partial_\gamma\partial_\mu\partial_\rho g_{\nu\sigma} - (\mu\leftrightarrow\nu)\right)\\ &= \frac{1}{2}\left(\partial_\gamma\partial_\mu\partial_\sigma g_{\rho\nu} - \partial_\gamma\partial_\mu\partial_\rho g_{\nu\sigma} - \partial_\gamma\partial_\nu\partial_\sigma g_{\rho\mu} + \partial_\gamma\partial_\nu\partial_\rho g_{\mu\sigma}\right).\end{aligned} \tag{9}$$

We then find that the sum of cyclic permutations of the three first indices of this tensor vanishes:

$$\nabla_\gamma R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\gamma\mu\nu} + \nabla_\sigma R_{\gamma\rho\mu\nu} = 0. \tag{10}$$

This was derived in a LICS, but, being a tensorial identity, it holds in any coordinate system.

Let us contract this identity on the 3rd and 5th index, to get

$$0 = \nabla_\gamma R_{\rho\sigma\mu}{}^\sigma + \nabla_\rho R_{\sigma\gamma\mu}{}^\sigma + \nabla_\sigma R_{\gamma\rho\mu}{}^\sigma = \nabla_\gamma R_{\rho\mu} - \nabla_\rho R_{\gamma\mu} + \nabla^\sigma R_{\gamma\rho\mu\sigma}, \tag{11}$$

where $R_{\alpha\beta}$ is the Ricci tensor. Let us now contract on γ and μ, and obtain

$$0 = \nabla^\gamma R_{\rho\gamma} - \nabla_\rho R + \nabla^\sigma R_{\rho\sigma} = 2\nabla^\gamma\left(R_{\rho\gamma} - \frac{1}{2}g_{\rho\gamma}R\right) = 2\nabla^\gamma G_{\rho\gamma}, \tag{12}$$

where R is the Ricci scalar and $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}R\ g_{\alpha\beta}$ is the Einstein tensor. So to summarize, the **contracted Bianchi identity** implies that the Einstein tensor is divergence-free:

$$\nabla_\mu G^{\mu\nu} = 0. \tag{13}$$

Newtonian limit and Einstein's field equation – The geodesic equation is

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\ \mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0. \tag{14}$$

Let us rewrite this in terms of the coordinate time t:

$$\begin{aligned}\frac{d^2x^k}{dt^2} &= \frac{d\tau}{dt}\frac{d}{d\tau}\left(\frac{1}{dt/d\tau}\frac{dx^k}{d\tau}\right) = \left(\frac{d\tau}{dt}\right)^2\left(\frac{d^2x^k}{d\tau^2} - \frac{d^2t}{d\tau^2}\frac{dx^k}{dt}\right) = -\Gamma^k_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} + \Gamma^0_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{dx^k}{dt}\\ &= -\Gamma^k_{00} - 2\Gamma^k_{0i}v^i - \Gamma^k_{ij}v^iv^j + \left(\Gamma^0_{00} + 2\Gamma^0_{0i}v^i + \Gamma^0_{ij}v^iv^j\right)v^k.\end{aligned} \tag{15}$$

Let us consider the Newtonian limit of GR, i.e. (*i*) small velocities $v \ll 1$, (*ii*) nearly flat spacetime, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},\ \ |h_{\mu\nu}| \ll 1$, and (*iii*), quasi-stationary spacetime, i.e. $\partial_t h_{\mu\nu} \ll \partial_x h_{\mu\nu}$. The small-velocity condition implies

$$\frac{d^2x^k}{dt^2} = -\Gamma^k_{00} + \mathcal{O}(v). \tag{16}$$

The Christoffel symbol is

$$\Gamma^k_{00} = \frac{1}{2}g^{k\mu}\left(2g_{0\mu,0} - g_{00,\mu}\right) = -\frac{1}{2}h_{00,k} + \mathcal{O}(\partial_t h, h\partial_k h). \tag{17}$$

To match the Newtonian equation $d^2x^k/dt^2 = -\nabla_k\phi$, we must have $\phi = -\frac{1}{2}h_{00}$.

Since the Christoffel symbols are of order gradients of the metric perturbations ∂h, the Riemann tensor is given by

$$R^\lambda{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\lambda{}_{\nu\sigma} - \partial_\nu\Gamma^\lambda{}_{\mu\sigma} + \mathcal{O}(\partial h)^2. \tag{18}$$

This implies that

$$R^i{}_{0j0} = \partial_j\Gamma^i{}_{00} + \mathcal{O}(\partial_t\Gamma) + \mathcal{O}(\partial h)^2 = -\frac{1}{2}h_{00,ij} = \partial_i\partial_j\phi. \tag{19}$$

Hence, the Ricci tensor has 00 component

$$R_{00} = R^i{}_{0i0} + R^0{}_{000} = R^i{}_{0i0} = \nabla^2\phi. \tag{20}$$

So Poisson's equation $\nabla^2\phi = 4\pi G\rho$ can be rewritten as $R_{00} = 4\pi G T_{00}$ – remember that T^{00} is the energy density, and that $T_{00} \approx T^{00}$ since $g_{\mu\nu} \approx \eta_{\mu\nu}$. It is tempting to "lift" this equation to a tensorial equality, but, while the stress-energy tensor is divergence-free, it is not the case for the Ricci tensor, so that's not the full story.

Define $\tilde{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\, T^\lambda_{\ \lambda}$, the trace-reversed stress-energy tensor. In the non-relativistic limit, $T_{ij} \ll T_{0i} \ll T_{00}$, so $T^\lambda_{\ \lambda} \approx -T_{00}$. Hence $\tilde{T}_{00} = \frac{1}{2}T_{00}$ in the non-relativistic limit. So we can also rewrite Poisson's equation as $R_{00} = 8\pi G\tilde{T}_{00}$. If we now "lift" this equation to become a tensorial equality, we get $R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\, T)$, or, equivalently (reversing the trace),

$$G_{\mu\nu} = 8\pi G\; T_{\mu\nu}. \tag{21}$$

From the contracted Bianchi identity $\nabla_\mu G^{\mu\nu}$, this equation is consistent with the conservation of stress-energy tensor. This is **Einstein's field equation**, which determines the spacetime curvature given the matter content.