General Relativity Fall 2018
Lecture 8: The Riemann tensor – continued

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Recap: definition of Riemann and symmetries – The Riemann tensor is defined through

$$\nabla_\mu \nabla_\nu V^\sigma - \nabla_\nu \nabla_\mu V^\sigma = R^\sigma{}_{\lambda\mu\nu} V^\lambda. \tag{1}$$

In a coordinate basis, the components of the Riemann tensor are given by

$$R^\sigma{}_{\lambda\mu\nu} = 2\left(\partial_{[\mu}\Gamma^\sigma_{\nu]\lambda} + \Gamma^\sigma_{\rho[\mu}\Gamma^\rho_{\nu]\lambda}\right). \tag{2}$$

Riemann = 0 ⇔ spacetime is flat – We saw that the change of a vector upon parallel transport around a small closed loop is proportional to the Riemann tensor times the area of the loop. This can equivalently be rephrased as follows: the difference in a vector parallel transported along two different paths is proportional to the Riemann tensor times the area of the loop enclosed by the two paths.

Therefore, if the Riemann tensor vanishes everywhere, *parallel transport is independent of the path* (assuming the manifold is simply connected). This holds even for a finite path. To prove this, subdivide a finite loop of area A into N loops of area (A/N). If Riemann vanishes, then the difference in parallel transport per loop (for small enough loop) scales as $(A/N)^\alpha$, where the index $\alpha > 1$. Hence the total error on a finite path scales as $N(A/N)^\alpha \propto A^\alpha/N^{\alpha-1}$ Taking $N \to \infty$, we see that this goes to zero.

Suppose Riemann vanishes. At a given point p, find an orthonormal basis of the dual space, $\{w^{(\mu)}\}$ (i.e. a basis of one-forms), such that $g = \eta_{\mu\nu} w^{(\mu)} \otimes w^{(\nu)}$ at p. This is always possible. Now parallel transport the $w^{(\mu)}$ everywhere in the manifold. This is a well-defined operation, as parallel transport is path-independent. So, for any vector field V^α, we have $V^\alpha \nabla_\alpha w^{(\mu)} = 0$. This implies that $\nabla w^{(\mu)} = 0$. Since g is metric compatible, the $n(n+1)/2$ *scalars* $g^{-1}(w^{(\mu)}, w^{(\nu)})$ have zero gradient everywhere, i.e. are constant and equal to $\eta^{\mu\nu}$, their value at the initial point p. This means that $g = \eta_{\mu\nu} w^{(\mu)} \otimes w^{(\nu)}$ everywhere.

Finally, let us show that we can find coordinates ξ^μ such that the associated dual basis is $d\xi^{(\mu)} = w^{(\mu)}$. To prove this, first pick som arbitray coordinate system $\{x^\mu\}$ with associated dual basis $dx^{(\mu)}$. Denote $w^{(\mu)}_\lambda$ the components of $w^{(\mu)}$ in this basis, i.e. $w^{(\mu)} = w^{(\mu)}_\lambda dx^{(\lambda)}$. Note that $\nabla_\alpha w^{(\mu)}_\beta = 0$ implies $\nabla_{[\alpha} w^{(\mu)}_{\beta]} = 0$, hence $\partial_\alpha w^{(\mu)}_\beta = \partial_\beta w^{(\mu)}_\alpha$. This implies that the the equations $\partial \xi^\mu/\partial x^\lambda = w^{(\mu)}_\lambda$ have solutions. The one-forms associated with the coordinates $\{\xi^\mu\}$ are then just $d\xi^{(\mu)} = (\partial\xi^\mu/\partial x^\lambda) d\xi^{(\lambda)} = w^{(\mu)}_\lambda d\xi^{(\lambda)} = w^{(\mu)}$, by definition. Even if the coordinates $\{x^\mu\}$ only cover a fraction of the manifold, we may solve $\partial\xi^\mu/\partial x^\lambda = w^{(\mu)}_\lambda$ in whatever coordinates are available: this equation is self consistent between different sets of coordinates (but remember that the functions ξ^λ are fixed, we are only talking about changing x^μ).

To conclude, we found coordinates in which $g = \eta_{\mu\nu} d\xi^\mu d\xi^\nu$, i.e. the vanishing of Riemann implies that spacetime is flat.

Commutator: definition – Given two vector fields X, Y we define their commutator $[X, Y]$ as the new vector field whose action on scalar functions is

$$[X, Y](f) \equiv X(Y(f)) - Y(X(f)). \tag{3}$$

To show that this is indeed a vector field, we must demonstrate that it is a linear operator on functions (that should be obvious), and that it satisfies Leibniz' rule:

$$\begin{aligned}[X, Y](fg) &= X(Y(fg)) - Y(X(fg)) = X(fY(g) + gY(f)) - Y(fX(g) + gX(f)) \\ &= f\left(X(Y(g)) - Y(X(g))\right) + g\left(X(Y(f)) - Y(X(f))\right) = f[X, Y](g) + g[X, Y](f).\end{aligned} \tag{4}$$

Let us pick a coordinate system and associated basis" $X(f) = X^\mu \partial_\mu f$, so that

$$[X, Y](f) = X^\nu \partial_\nu \left(Y^\mu \partial_\mu f\right) - Y^\nu \partial_\nu \left(X^\mu \partial_\mu f\right) = X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu. \tag{5}$$

Now, let us compute the components of $\nabla_X Y - \nabla_Y X$ in a coordinate basis:

$$\nabla_X Y^\mu - \nabla_Y X^\mu = X^\nu \left(\partial_\nu Y^\mu + \Gamma^\mu_{\nu\sigma} Y^\sigma\right) - Y^\nu \left(\nabla_\nu X^\mu + \Gamma^\mu_{\nu\sigma} X^\sigma\right) = X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu, \tag{6}$$

where we used the symmetry of the Christoffel symbol. Hence, we see that we can also write the commutator as

$$[X, Y] = \nabla_X Y - \nabla_Y X. \tag{7}$$

If X and Y are vectors of a coordinate basis, e.g. $X = \partial_{(1)}$ and $Y = \partial_{(2)}$, they have constant components in that coordinate basis (δ^μ_1 and δ^μ_2, respectively), so their commutator vanishes. **Theorem**: the commutator of coordinate basis vector fields vanishes.

Riemann in terms of the commutator – Let us re-express the following vector field:

$$\begin{aligned}(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z)^\alpha &= X^\beta \nabla_\beta (Y^\gamma \nabla_\gamma Z^\alpha) - Y^\beta \nabla_\beta (X^\gamma \nabla_\gamma Z^\alpha)\\ &= (X^\beta \nabla_\beta Y^\gamma)(\nabla_\gamma Z^\alpha) + Y^\gamma X^\beta \nabla_\beta (\nabla_\gamma Z^\alpha) - (Y^\beta \nabla_\beta X^\gamma)(\nabla_\gamma Z^\alpha) + X^\gamma Y^\beta \nabla_\beta (\nabla_\gamma Z^\alpha)\\ &= (\nabla_X Y - \nabla_Y X)^\gamma \nabla_\gamma Z^\alpha + Y^\gamma X^\beta \left(\nabla_\beta \nabla_\gamma Z^\alpha - \nabla_\gamma \nabla_\beta Z^\alpha\right)\\ &= \nabla_{[X,Y]} Z^\alpha + R^\alpha{}_{\delta\beta\gamma} Z^\delta X^\beta Y^\gamma. \end{aligned} \tag{8}$$

Therefore, we may also rewrite the Riemann tensor as

$$\text{Riemann}(-, Z, X, Y) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{9}$$

Geodesic deviation – Consider a family of geodesics, parametrized by two coordinates: t, along each geodesic, and s, from one geodesic to the next. We complete the coordinate system as needed, and define the coordinate basis vactor fields. In particular, $T \equiv \partial/\partial t$ is the tangent to the geodesics. We now ask, how does the vector $S \equiv \partial/\partial s$ evolve along the geodesics:

$$\begin{aligned}\nabla_T(\nabla_T S) &= \nabla_T(\nabla_S T) \quad \text{(Commutator vanishes for basis vector fields)}\\ &= \nabla_T \nabla_S T - \nabla_S \nabla_T T \quad (\nabla_T T = 0 \text{ for geodesics with tangent vector } T)\\ &= \text{Riemann}(-, T, T, S) \quad \text{(using relation above, and } [T, S] = 0). \end{aligned} \tag{10}$$

In other words, we found that

$$\nabla_T \nabla_T S^\alpha = R^\alpha{}_{\beta\gamma\delta} T^\beta T^\gamma S^\delta. \tag{11}$$

This can be interpreted as the deviation of geodesics due to tidal fields, encoded by the Riemann tensor. This is the standard derivation, but I'll try to give a more physical (and satisfactory) derivation later on.

Number of independent components of Riemann – Let us first recall the symmetries of the fully covariant Riemann tensor $R_{\delta\lambda\mu\nu} \equiv g_{\delta\sigma} R^\sigma{}_{\lambda\mu\nu}$:

$$R_{\delta\lambda\nu\mu} = -R_{\delta\lambda\mu\nu}, \tag{12}$$

$$R_{\lambda\delta\mu\nu} = -R_{\delta\lambda\mu\nu}, \tag{13}$$

$$R_{\mu\nu\delta\lambda} = R_{\delta\lambda\mu\nu}, \tag{14}$$

$$R_{\delta\lambda\mu\nu} + R_{\delta\mu\nu\lambda} + R_{\delta\nu\lambda\mu} = 0. \tag{15}$$

The number of independent components in each antisymmetric pair of indices is $N = n(n-1)/2$. If only the first 3 symmetry conditions were satified, we would have $N(N+1)/2$ independent components. We now want to know how many *independent constraints* the fourth symmetry property provides. Define $R_{\delta\{\lambda\mu\nu\}} \equiv R_{\delta\lambda\mu\nu} + R_{\delta\mu\nu\lambda} + R_{\delta\nu\lambda\mu}$. If any two indices are the same, then this quantity automatically vanishes due to the antisymmetry of Riemann in its first two and last two indices (check it!). Therefore, the fourth symmetry property adds information only if all 4 indices are different. This means that, in dimension n, it provides $C^4_n = n(n-1)(n-2)(n-3)/24$ independent conditions. Therefore, the total number of independent components of Riemann is

$$\frac{1}{2}\frac{n(n-1)}{2}\left(\frac{n(n-1)}{2} + 1\right) - \frac{n(n-1)(n-2)(n-3)}{24} = \frac{n^2(n^2-1)}{12}. \tag{16}$$

This is precisely the difference between the number of second derivatives of the metric tensor $\partial_\delta \partial_\sigma g_{\mu\nu}$ and the number of third derivatives of coordinates $\partial^3 x^\delta / \partial x^\lambda \partial x^\mu \partial x^\nu$, i.e

$$\left(\frac{n(n+1)}{2}\right)^2 - n\frac{n(n+1)(n+2)}{6} = \frac{n^2(n^2-1)}{12}. \tag{17}$$

In other words, the independent components of the Riemann tensor can be thought of as the $n^2(n^2-1)/12$ second derivatives of the metric tensor that *cannot* be set to zero by coordinate transformations.

Ricci tensor and scalar – We may define the Ricci tensor and scalar, respectively, as $R_{\alpha\beta} \equiv R^\gamma{}_{\alpha\gamma\beta}$ and $R \equiv R^\alpha{}_\alpha$ (it is customary to use the same letter for all...). The Ricci tensor is symmetric.

Curvature as a function of dimension, Weyl tensor

• In dimension $n = 1$, the Riemann tensor has 0 independent components, i.e. vanishes everywhere. There is no **intrinsic curvature** in 1-dimension. An ant walking on a line does not feel curvature (even if the line has an extrinsic curvature if seen as embedded in $\mathbb{R}^2$).

• In dimension $n = 2$, the Riemann tensor has one independent component. It is therefore entirely determined by the Ricci scalar, or scalar curvature: $R_{\alpha\beta\gamma\delta} = R\ g_{\alpha[\gamma} g_{\delta]\beta}$.

• In dimension $n = 3$, the Riemann tensor has 6 independent components, just as many as the symmetric Ricci tensor. The Riemann tensor is entirely determined by the 6 independent components of the Ricci tensor:

$$R_{\alpha\beta\gamma\delta} = (g_{\alpha\gamma} R_{\beta\delta} - g_{\alpha\delta} R_{\beta\gamma} - g_{\beta\gamma} R_{\alpha\delta} + g_{\beta\delta} R_{\alpha\gamma}) + \frac{R}{2}\left(g_{\beta\gamma} g_{\alpha\delta} - g_{\alpha\gamma} g_{\beta\delta}\right). \tag{18}$$

One can check that this expression gives the Ricci tensor upon contraction.

• Finally, in $n \geq 4$, the Riemann tensor contains more information than there is in Ricci: we define the **Weyl tensor** $C_{\alpha\beta\gamma\delta}$ as

$$R_{\alpha\beta\gamma\delta} = \frac{1}{n-2}\left(g_{\alpha\gamma} R_{\beta\delta} - g_{\alpha\delta} R_{\beta\gamma} - g_{\beta\gamma} R_{\alpha\delta} + g_{\beta\delta} R_{\alpha\gamma}\right) + \frac{R}{(n-1)(n-2)}\left(g_{\beta\gamma} g_{\alpha\delta} - g_{\alpha\gamma} g_{\beta\delta}\right) + C_{\alpha\beta\gamma\delta}. \tag{19}$$

The first two pieces have the correct symmetries, and, when contracted, give the Ricci tensor and scalar. The remainder $C_{\alpha\beta\gamma\delta}$ has the same symmetries as the Riemann tensor, and is in addition trace-free, $C^\alpha{}_{\beta\alpha\gamma} = 0$. This means that it has

$$\frac{n^2(n^2-1)}{12} - \frac{n(n+1)}{2} = \frac{n(n+1)}{2}\left[\frac{n(n-1)}{6} - 1\right] \tag{20}$$

independent components. In dimenions 4, this is 10 independent components.

Einstein tensor – The Einstein tensor is the trace-reverse of the Ricci tensor: $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}$.