General Relativity Fall 2018
Lecture 10: Integration, Einstein-Hilbert action

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(Dated: October 3, 2018)

HW comment: $T^\sigma{}_{\mu\nu}$ antisym in $\mu\nu$ does NOT imply that $T_{\mu\nu}{}^\sigma$ is antisym in $\mu\nu$.

Volume element – Consider a LICS with primed coordinates. The 4-volume element is $d\mathcal{V} = d^4x' = dx^{0'}dx^{1'}dx^{2'}dx^{3'}$. If we change coordinates, we have

$$d^4x' = \left|\det\left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right)\right| d^4x. \tag{1}$$

Now, the metric components change as

$$g_{\mu\nu} = \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^{\nu'}}{\partial x^\nu} g_{\mu'\nu'} = \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^{\nu'}}{\partial x^\nu}\eta_{\mu'\nu'}, \tag{2}$$

since the metric components are $\eta_{\mu'\nu'}$ in the LICS. Seing this as a matrix operation and taking the determinant, we find

$$\det(g_{\mu\nu}) = -\det\left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right)^2. \tag{3}$$

Hence, we find that the 4-volume element is

$$d\mathcal{V} = \sqrt{-\det(g_{\mu\nu})}d^4x \equiv \sqrt{-g}\ d^4x \equiv \sqrt{|g|}\ d^4x. \tag{4}$$

The integral of a scalar function f is well defined: given any coordinate system (even if only defined locally):

$$\int f\ d\mathcal{V} = \int f\ \sqrt{|g|}\ d^4x. \tag{5}$$

We can only define the integral of a scalar function. The integral of a vector or tensor field is meaningless in curved spacetime. Think of the integral as a sum. To sum vectors, you need them to belong to the same vector space. There is no common vector space in curved spacetime.

Only in flat spacetime can we define such integrals. First parallel-transport the vector field to a single point of spacetime (it doesn't matter which one). This operation is independent of the path in flat spacetime, hence well defined. Then sum these components to perform the integral.

Covariant divergence– The covariant divergence of a vector field is

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\nu}V^\nu. \tag{6}$$

The contracted Christoffel symbol is

$$\Gamma^\mu_{\mu\nu} = \frac{1}{2}g^{\mu\lambda}\left(\partial_\nu g_{\mu\lambda} + \partial_\mu g_{\nu\lambda} - \partial_\lambda g_{\mu\nu}\right) = \frac{1}{2}g^{\mu\lambda}\partial_\nu g_{\mu\lambda} + g^{\mu\lambda}\partial_{[\lambda}g_{\mu]\nu} = \frac{1}{2}g^{\mu\lambda}\partial_\nu g_{\mu\lambda}. \tag{7}$$

Let us now recall that the determinant and inverse of a matrix (here, the metric $g_{\mu\nu}$) can be expressed in terms of the comatrix $C_{\mu\nu}$, which is the matrix made of the determinants of the $(n-1)\times(n-1)$ submatrices obtained by deleting row μ and column ν (and then multiplying by $(-1)^{\mu+\nu)}$):

$$g \equiv \det(g_{\mu\nu}) = \sum_\mu g_{\mu\nu}C_{\mu\nu}, \quad \text{for any fixed } \nu \text{ (not summed over!)}, \tag{8}$$

$$g^{\mu\nu} = \frac{1}{g}C_{\mu\nu}, \tag{9}$$

Therefore, we get

$$\frac{\partial_\nu \sqrt{|g|}}{\sqrt{|g|}} = \frac{1}{2}\frac{1}{g}\frac{\partial g}{\partial g_{\mu\lambda}}\partial_\nu g_{\mu\lambda} = \frac{1}{2}\frac{1}{g}C_{\mu\lambda}\partial_\nu g_{\mu\lambda} = \frac{1}{2}g^{\mu\lambda}\partial_\nu g_{\mu\lambda} = \Gamma^\mu_{\mu\nu}. \tag{10}$$

So, we find

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \frac{\partial_\nu \sqrt{|g|}}{\sqrt{|g|}}V^\nu = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}V^\mu). \tag{11}$$

Stokes' theorem – First consider a 4-volume $\mathcal{V}$ covered by a single coordinate system. Since $\nabla_\mu V^\mu$ is a scalar field, we may compute its integral over $\mathcal{V}$:

$$\int_\mathcal{V} \nabla_\mu V^\mu \; d\mathcal{V} = \int_\mathcal{V} \frac{\partial_\mu(\sqrt{|g|}V^\mu)}{\sqrt{|g|}} \; \sqrt{|g|}d^4x = \int_\mathcal{V} \partial_\mu(\sqrt{|g|}V^\mu)d^4x. \tag{12}$$

This is just a normal integral in a volume of $\mathbb{R}^4$, for which we can use Stokes' theorem. We denote by $\partial\mathcal{V}$ the boundary of $\mathcal{V}$. We denote by n^μ the vector normal to the surface, with squared norm ± 1, pointing outwards if it is spacelike, and inwards if it is timelike. You can check that this is is required to get back Stokes' theorem – to convince yourself, use coordinates in which the normal to the surface is a coordinate basis vector $\partial_{(\mu)}$ and use the usual Stokes theorem.

By definition, this vector is orthogonal to the vectors spanning the tangent space of $\partial\mathcal{V}$, so the metric can be rewritten as $g_{\mu\nu} = \pm n_\mu n_\nu + g^{(3)}_{\mu\nu}$, where $g^{(3)}_{\mu\nu} n^\mu = 0$: $g^{(3)}$ is the induced metric on the 3-dimensional manifold $\partial\mathcal{V}$. We then have $\det(g_{\mu\nu}) = \pm\det(g^{(3)}_{\mu\nu})$, i.e. $|g| = |g^{(3)}|$. We then get

$$\int_\mathcal{V} \nabla_\mu V^\mu \; d\mathcal{V} = \int_{\partial\mathcal{V}} n_\mu \sqrt{|g|}V^\mu d^3x = \int_{\partial\mathcal{V}} n_\mu V^\mu \sqrt{|g^{(3)}|}d^3x. \tag{13}$$

We recognize the last two factors as the frame-invariant volume element on the boundary. This last integral is therefore all coordinate-independent, as it should.

Now, if we need to use multiple charts to cover $\mathcal{V}$, we can subdivide it into sub-volumes each with a single coordinate system, and apply Stoke's theorem there. The integrals on interior surfaces cancel out, and we are left with the boundary of the full volume.

Action formulation of general relativity – The Einstein field equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$ (from now we use geometric units where $G = c = 1$) can be derived from an action principle. We want to build an overall action of the form

$$S = S_{\text{gravity}} + S_M \equiv \int d^4x\sqrt{|g|}\mathcal{L}_{\text{gravity}} + \int d^4x\sqrt{|g|}\mathcal{L}_M, \tag{14}$$

where S_M is the matter action, and S_{gravity} only involves the metric tensor, and $\mathcal{L}$ are the corresponding frame-invariant Lagrangian densities. The most natural scalar function of curvature that comes to mind is the Ricci scalar, so we may anticipate that $\mathcal{L}_{\text{gravity}} \propto R$. The idea then, is to recover the Eintein field equation by extremizing the action with respect to variations in the metric tensor and in the matter fields (whatever they are).

To compute these variations, pick a coordinate system. We want to compute the variation of the action upon variation of the metric field by $\delta g_{\mu\nu}$:

$$\begin{aligned}\delta(\sqrt{-g}R) &= R\delta(\sqrt{|g|}) + \sqrt{|g|}\;\delta R = \sqrt{|g|}\left(R\frac{\delta(\sqrt{|g|})}{\sqrt{|g|}} + \delta R\right) = \sqrt{|g|}\left(\frac{1}{2}g^{\mu\nu}\delta g_{\mu\nu} + \delta R\right)\\ &= \sqrt{|g|}\left(-\frac{1}{2}Rg_{\mu\nu}\delta g^{\mu\nu}R + \delta R\right),\end{aligned} \tag{15}$$

where we used $\delta(g_{\mu\nu}g^{\mu\nu}) = 0 = g_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta g_{\mu\nu}$. Now we have

$$R = g^{\mu\nu}R_{\mu\nu} \quad \Rightarrow \delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}. \tag{16}$$

This implies

$$\delta(\sqrt{-g}R) = \sqrt{|g|}\left(G_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right). \tag{17}$$

Let us now compute the variation of the Riemann tensor under variation of the metric. To do so, recall that the Riemann tensor is defined in terms of the commutator of the metric-compatible covariant derivative,

$$R^{\lambda}{}_{\sigma\mu\nu}V^{\sigma} \equiv \nabla_{\mu}\nabla_{\nu}V^{\lambda} - \nabla_{\nu}\nabla_{\mu}V^{\lambda}. \tag{18}$$

Let us denote by $\tilde{\nabla}$ the unique metric-compatible (and torsion-free) covariant derivative associated with $g_{\mu\nu} + \delta g_{\mu\nu}$, we then have

$$\delta R^{\lambda}{}_{\sigma\mu\nu}V^{\sigma} = \tilde{\nabla}_{\mu}\tilde{\nabla}_{\nu}V^{\lambda} - \nabla_{\mu}\nabla_{\nu}V^{\lambda} - (\mu \leftrightarrow \nu). \tag{19}$$

We denote by $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ the associated Christoffel symbols, and $\delta\Gamma = \tilde{\Gamma} - \Gamma$ the difference relative to those associated with $g_{\mu\nu}$. We then have, for any vector,

$$\tilde{\nabla}_{\nu}V^{\lambda} = \nabla_{\nu}V^{\lambda} + \delta\Gamma^{\lambda}_{\nu\sigma}V^{\sigma}, \tag{20}$$

where the partial derivatives cancel out. Because the left-hand side is a tensor and so is V^{σ}, we conclude that $\delta\Gamma$ is a rank $(1, 2)$ tensor (even if each coefficient individually is not!), symmetric in its lower two indices. Similarly, we have for any rank (1, 1) tensor,

$$\tilde{\nabla}_{\mu}X^{\lambda}{}_{\nu} = \nabla_{\mu}X^{\lambda}{}_{\nu} + \delta\Gamma^{\lambda}{}_{\mu\sigma}X^{\sigma}{}_{\nu} - \delta\Gamma^{\sigma}_{\mu\nu}X^{\lambda}{}_{\sigma} \tag{21}$$

We then have

$$\begin{aligned}
\delta R^{\lambda}{}_{\sigma\mu\nu}V^{\sigma} &= \tilde{\nabla}_{\mu}\nabla_{\nu}V^{\lambda} - \nabla_{\mu}\nabla_{\nu}V^{\lambda} + \tilde{\nabla}_{\mu}(\delta\Gamma^{\lambda}_{\nu\sigma}V^{\sigma}) - (\mu \leftrightarrow \nu) \\
&= \delta\Gamma^{\lambda}{}_{\mu\sigma}\nabla_{\nu}V^{\sigma} - \delta\Gamma^{\sigma}_{\mu\nu}\nabla_{\sigma}V^{\lambda} + \nabla_{\mu}(\delta\Gamma^{\lambda}_{\nu\sigma}V^{\sigma}) + \mathcal{O}(\delta\Gamma^2) - (\mu \leftrightarrow \nu) \\
&= \delta\Gamma^{\lambda}{}_{\mu\sigma}\nabla_{\nu}V^{\sigma} - \delta\Gamma^{\sigma}_{\mu\nu}\nabla_{\sigma}V^{\lambda} + \nabla_{\mu}(\delta\Gamma^{\lambda}_{\nu\sigma})V^{\sigma} + \delta\Gamma^{\lambda}_{\nu\sigma}\nabla_{\mu}V^{\sigma} + \mathcal{O}(\delta\Gamma^2) - (\mu \leftrightarrow \nu)
\end{aligned} \tag{22}$$

The second term is symmetric in (μ, nu) so cancels upon antisymmetrization. The sum of the first and fourth terms is also symmetric in (μ, ν), and cancels. We are then left with

$$\delta R^{\lambda}{}_{\sigma\mu\nu} = \nabla_{\mu}(\delta\Gamma^{\lambda}{}_{\nu\sigma}) - \nabla_{\nu}(\delta\Gamma^{\lambda}{}_{\mu\sigma}). \tag{23}$$

This implies that

$$\delta R_{\sigma\nu} = \delta R^{\lambda}{}_{\sigma\lambda\nu} = \nabla_{\lambda}(\delta\Gamma^{\lambda}{}_{\nu\sigma}) - \nabla_{\nu}(\delta\Gamma^{\lambda}{}_{\lambda\sigma}), \tag{24}$$

and so

$$g^{\sigma\nu}\delta R_{\sigma\nu} = \nabla_{\lambda}(g^{\sigma\nu}\delta\Gamma^{\lambda}{}_{\nu\sigma}) - \nabla_{\nu}(g^{\sigma\nu}\delta\Gamma^{\lambda}{}_{\lambda\sigma}) = \nabla_{\mu}\left(g^{\sigma\nu}\delta\Gamma^{\mu}{}_{\nu\sigma} - g^{\sigma\mu}\delta\Gamma^{\lambda}{}_{\lambda\sigma}\right). \tag{25}$$

Hence, we find that, upon small variations of the metric, we have

$$\delta\int d^4x\sqrt{|g|}R = \int d^4x\sqrt{|g|}G_{\mu\nu}\delta g^{\mu\nu} + \int d^4x\sqrt{|g|}\nabla_{\mu}\left(g^{\sigma\nu}\delta\Gamma^{\mu}{}_{\nu\sigma} - g^{\sigma\mu}\delta\Gamma^{\lambda}{}_{\lambda\sigma}\right) = \int d^4x\sqrt{|g|}G_{\mu\nu}\delta g^{\mu\nu}, \tag{26}$$

since the last term is a total divergence and hence integrgates to zero provided metric perturbations (and their partial derivatives) vanish "at infinity".

The convention is to define the **Einstein-Hilbert action** with a prefactor of $1/(16\pi)$:

$$S = \frac{1}{16\pi}\int d^4x\sqrt{|g|}R + S_M, \tag{27}$$

and to *define* the stress-energy tensor of matter as

$$T_{\mu\nu} = -\frac{2}{\sqrt{|g|}}\frac{\delta S_M}{\delta g^{\mu\nu}} \tag{28}$$

By varying the total action with respect to the metric tensor, we see that we recover the Einstein Field Equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$. See e.g. Carroll for a couple of worked out examples (scalar field, electromagnetic field).