General Relativity Fall 2018
Lecture 11: Linearized gravity, Part I

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Einstein's field equations are fundamentally non-linear, and do not have analytic solutions except in very special cases: either spacetimes with a high degree of symmetry, or nearly flat spacetimes. In the next few weeks we will study the latter.

I. WARM-UP: ELECTROMAGNETISM

A. Gauge transformations; gauge-invariant variables

In flat spacetime (or in a locally inertial coordinate system) Maxwell's equations are

$$J^\nu = \partial_\mu F^{\nu\mu} \equiv \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \Box A^\nu - \partial^\nu \partial_\mu A^\mu, \qquad \Box \equiv \partial_\mu \partial^\mu, \tag{1}$$

where A^μ is the 4-vector potential, defined by $F_{\mu\nu} = 2\partial_{[\nu}A_{\mu]}$. A priori there are 4 degrees of freedom in this theory, the 4 components of A^μ. However, we can make the following **gauge transformations** without altering $F^{\mu\nu}$ hence any physical observables:

$$A^\mu \rightarrow A^\mu + \partial^\mu f, \qquad \text{i.e.} \quad \begin{cases} A^0 \rightarrow & A^0 - \partial_t f, \\ \vec{A} \rightarrow & \vec{A} + \vec{\nabla} f, \end{cases} \tag{2}$$

where f is an arbitrary function. Therefore not all 4 components of A^μ are physically meaningful. For instance, we can always chose a gauge in which $A^0 = 0$: given some initial A^0, just pick a function f such that $\partial_t f = A^0$.

Let us rewrite the 3-vector field $\vec{A}$ as the gradient of a scalar function and the curl of a divergence-free vector:

$$\vec{A} = \vec{\nabla}\psi + \vec{\nabla} \times \vec{V}, \qquad \vec{\nabla} \cdot \vec{V} = 0. \tag{3}$$

In Fourier space, these equations become

$$\vec{A} = i\vec{k}\psi + i\vec{k} \times \vec{V}, \qquad \vec{k} \cdot \vec{V} = 0. \tag{4}$$

The vector $\vec{V}$ is also said to be **transverse**, as it is transverse to the wavenumber in Fourier space.
It contains **2 degrees of freedom**, i.e. 2 independent functions orthogonal to $\vec{k}$ in Fourier space.

In terms of $\vec{A}$, these two fields are uniquely obtained from the Laplace equations (assuming appropriate boundary conditions at infinity),

$$\nabla^2 \psi = \vec{\nabla} \cdot \vec{A}, \qquad \nabla^2 \vec{V} = -\vec{\nabla} \times \vec{A}, \tag{5}$$

or, in Fourier space,

$$-k^2 \psi = i\vec{k} \cdot \vec{A}, \qquad -k^2 \vec{V} = -i\vec{k} \times \vec{A}. \tag{6}$$

So far we have simply re-organized the 3 degrees of freedom of $\vec{A}$ into a **scalar** function and a **transverse vector**. Just like A^0, the function ψ is invariant under spatial rotations, while $\vec{V}$ transforms as a vector under spatial rotations. So we rewrote our 4 initial degrees of freedom as 2 (2 scalars) + 2 (1 transverse vector).

Let us now look at how these components transform under gauge transformations. Clearly, the curl of $\vec{A}$ is unchanged by a gauge transformation (since it consists in adding a gradient). Hence **the transverse vector $\vec{V}$ is gauge-invariant**. We can also read off how ψ transforms: $\psi \rightarrow \psi + f$. Since we have $A^0 \rightarrow A^0 - \partial_t f$, we conclude that the combination

$$\phi \equiv A^0 + \partial_t \psi \tag{7}$$

is also gauge-invariant. We have therefore identified **3 gauge-invariant, physical degrees of freedom**: one scalar component, ϕ, and the transverse vector $\vec{V}$. The 4th degree of freedom in A^μ is unphysical and depends on an arbitrary choice of gauge. What this means in practice is that we can always pick a gauge in which either A^0 or ψ vanish, but we can never change the linear combination ϕ with a gauge transformation.

B. Maxwell's equations

Let us now look at Maxwell's equations in terms of these components:

$$J^0 = \Box A^0 - \partial^0\partial_\mu A^\mu = \nabla^2 A^0 + \partial_t\partial_i A^i = \nabla^2\left(A^0 + \partial_t\psi\right) = \nabla^2\phi, \qquad [\partial^0 = -\partial_0 = -\partial_t] \tag{8}$$

$$\vec{J} = \Box\vec{A} - \vec{\nabla}\left(\partial_j A^j + \partial_t A^0\right) = \Box\vec{A} - \vec{\nabla}\left(\nabla^2\psi + \partial_t A^0\right). \tag{9}$$

Taking the divergence of the spatial equation gives us

$$\vec{\nabla}\cdot\vec{J} = \nabla^2\left[\Box\psi - \nabla^2\psi - \partial_t A^0\right] = -\nabla^2\left[\partial_t^2\psi + \partial_t A^0\right] = -\partial_t\nabla^2\phi = -\partial_t J^0, \tag{10}$$

as expected from **conservation of charge** $\partial_\mu J^\mu = 0$. So the divergence part of the spatial equation is just the time derivative of the 0-equation, i.e., does not add any new information: if we require $J^0 = \nabla^2\phi$ at all times, then obviously their time derivatives must also be equal. This redundancy arises because the electromagnetic tensor satisfies $\partial_\mu\partial_\nu F^{\mu\nu} = 0$ (as it should so Maxwell's equations are consistent with charge conservation), so **only 3 out of the 4 Maxwell's equations are functionally independent**.

Let us now take the curl of the spatial component:

$$\Box\nabla^2\vec{V} = -\vec{\nabla}\times\vec{J}. \tag{11}$$

So Maxwell's equations **only determine the 3 gauge-invariant** (physical) degrees of freedom ϕ and $\vec{V}$. In order to fully specify A^μ, we need to additionally **impose a gauge condition**. We are free to choose whichever gauge condition is best adapted to the problem at hand (i.e. for which the equations look simpler). Two popular gauges in E&M are the **Coulomb gauge** $\vec{\nabla}\cdot\vec{A} = 0 \Rightarrow \psi = 0$, and the **Lorenz gauge** $\partial_\mu A^\mu = 0$**, i.e.** $\nabla^2\psi + \partial_t A^0 = 0$**.** [note, it is *Lorenz*, after Ludvig Lorenz, and not Lorentz, after Hendrik Lorentz; this is a common typo]. In the Lorenz gauge, Maxwell's equations simplify to $\Box A^\mu = J^\mu$. Note that all this works in flat spacetime: as we saw in the homework, this equation is modified in the presence of curvature.

A final point we will make is on the character of the time and spatial components of Maxwell's equations. The first equation, $\nabla^2\phi = J^0$, is a **constraint equation**: it does not contain any time derivative – if the time derivative in $\phi \equiv A^0 + \partial_t\psi$ bothers you, think that you can always pick a gauge (the Coulomb gauge) where $\psi = 0$.

On the other hand, the curl of the spatial equation is truly a **dynamical equation** for the gauge-invariant transverse vector $\vec{V}$. So electromagnetism has **2 dynamical degres of freedom**. In the absence of sources ($J^\mu = 0$), one can gauge away the scalar degrees of freedom: Maxwell's constraint equations already tell us that one linear combination must vanish (with proper boundary conditions), and we can choose a gauge where they both vanish (take for instance the Coulomb gauge $\psi = 0$). However, the transverse mode satisfies a homogeneous wave equation, which has non-vanishing wave solutions: these are the **electromagnetic waves, which can exist even in vacuum**.

II. LINEARIZED GRAVITY

A. Starting point: nearly globally-Lorentz metric; gauge transformations

We know that one can always find coordinate systems in which the metric is nearly Minkowski *locally* (these are locally inertial coordinate systems). Here we assume that there exists some coordinate system in which the metric is **nearly Minkowski globally**, i.e., that throughout spacetime,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| = \mathcal{O}(\epsilon) \ll 1. \tag{12}$$

The metric perturbation $h_{\mu\nu}$ **is not a generally covariant tensor** (the full metric $g_{\mu\nu}$ is), but does transform as a tensor under local Lorentz transformations. Indeed, under a Lorentz transformation,

$$g'_{ab} = \Lambda^\alpha{}_a\Lambda^\beta{}_b\, g_{\alpha\beta} = \Lambda^\alpha{}_a\Lambda^\beta{}_b(\eta_{\alpha\beta} + h_{\alpha\beta}) = \eta_{ab} + \Lambda^\alpha{}_a\Lambda^\beta{}_b h_{\alpha\beta}, \tag{13}$$

where we used the fact that the components of the Lorentz metric $\eta_{\mu\nu}$ are unchanged by Lorentz transformations (which is the defining property of Lorentz transformations!). Therefore, we find that, under a Lorentz transformation, $h'_{ab} = \Lambda^\alpha{}_a\Lambda^\beta{}_b h_{\alpha\beta}$, i.e. $h_{\mu\nu}$ **transforms like a special-relativity tensor (a Lorentz-tensor)**.

The next thing to ask is how $h_{\mu\nu}$ transforms under a more general (yet small) change of coordinates,

$$x^\mu(p) = y^\mu(p) + \xi^\mu(p), \qquad \xi^\lambda_{,\mu} = \mathcal{O}(\epsilon) \ll 1, \tag{14}$$

where p is the space-time point on the manifold. Then we know that in the y-coordinate basis, the metric coefficients at a given space-time point are such that

$$g^{(y)}_{\mu\nu} = g^{(x)}_{\lambda\sigma}\frac{\partial x^\lambda}{\partial y^\mu}\frac{\partial x^\sigma}{\partial y^\nu} = \left(\delta^\lambda_\mu + \xi^\lambda{}_{,\mu}\right)\left(\delta^\sigma_\nu + \xi^\sigma{}_{,\nu}\right)\left(\eta_{\lambda\sigma} + h_{\lambda\sigma}\right). \tag{15}$$

Now, provided the derivatives of ξ are small, $\xi^\alpha_{,\mu} = \mathcal{O}(\epsilon) \ll 1$, we obtain

$$g^{(y)}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \eta_{\mu\lambda}\xi^\lambda{}_{,\nu} + \eta_{\nu\lambda}\xi^\lambda{}_{,\mu} + \mathcal{O}(\epsilon^2) \equiv \eta_{\mu\nu} + h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} + \mathcal{O}(\epsilon^2) \equiv \eta_{\mu\nu} + h^{(y)}_{\mu\nu}. \tag{16}$$

Hence we find

$$h^{(y)}_{\mu\nu} = h^{(x)}_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} + \mathcal{O}(\epsilon^2). \tag{17}$$

Now this is expressed at the same space-time point, i.e.

$$\begin{aligned} h^{(y)}_{\mu\nu}(y^\alpha) &= h^{(x)}_{\mu\nu}(x^\alpha = y^\alpha + \xi^\alpha) + \xi_{\mu,\nu} + \xi_{\nu,\mu} + \mathcal{O}(\epsilon^2) = h^{(x)}_{\mu\nu}(y^\alpha) + \xi^\alpha\partial_\alpha h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} + \mathcal{O}(\epsilon^2) \\ &= h^{(x)}_{\mu\nu}(y^\alpha) + \xi_{\mu,\nu} + \xi_{\nu,\mu} + \mathcal{O}(\epsilon^2). \end{aligned} \tag{18}$$

Here we have re-expressed things **at the same coordinate values**, rather than at the same spacetime point. In this context, the two things are the same at linear order; however, when perturbing about a non-flat background (for instance, in cosmological perturbation theory), one needs to include the extra term in the second line.

Now, the Riemann tensor vanishes for a flat metric, and as a consequence must be of order ϵ. As a consequence, a small change of coordinates leaves the **components of the Riemann tensor unchanged at linear order**:

$$R'_{abcd} = \frac{\partial x_\alpha}{\partial x'_a}\frac{\partial x_\beta}{\partial x'_b}\frac{\partial x_\gamma}{\partial x'_c}\frac{\partial x_\delta}{\partial x'_d}R_{\alpha\beta\gamma\delta} = R_{abcd}\left[1 + \mathcal{O}(\epsilon)\right]. \tag{19}$$

Again, this holds at the same spacetime point but remains true at the same coordinate values.

To conclude, we can view the metric perturbation $h_{\mu\nu}$ **as a special-relativity tensor** (i.e. transforms as a tensor under Lorentz transformations), and the **small coordinate transformation as a gauge transformation**. Under such gauge transformation, the equivalent of $A_\mu \to A_\mu + \partial_\mu f$ in electromagnetism is

$$h_{\mu\nu} \to h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} = h_{\mu\nu} + 2\xi_{(\mu,\nu)}. \tag{20}$$

Such a gauge transformation does not affect any measurable quantity at linear order. It does not change the components of the Riemann tensor (at linear order), which determines tidal fields, and is the equivalent of $F_{\mu\nu}$ in electromagnetism.

B. Scalar-vector-tensor (SVT) decomposition

Just like for the vector potential A^μ, not all components of $h_{\mu\nu}$ have physical meaning (we may always set some to zero with appropriate gauge choices). To see this, the next thing is to generalize the scalar + transverse-vector decomposition of A^μ to $h_{\mu\nu}$. Here we use "scalar", "vector" and "tensor" to characterize **transformation properties under spatial rotations**.

- h_{00} is a scalar under spatial rotations.
- We can think of h_{0i} as a 3-vector

$$\vec{h}_0 \equiv (h_{01}, h_{02}, h_{03}), \tag{21}$$

and write is as the gradient of a scalar field and the curl of a transverse vector field, i.e. just like for A_i, we write

$$\vec{h}_0 = \vec{\nabla}\chi + \vec{\nabla}\times\vec{V}, \qquad \vec{\nabla}\cdot\vec{V} = 0. \tag{22}$$

We are left with the 6 components of h_{ij}. The SVT decomposition is best done in Fourier space:

$$h_{ij} = \frac{1}{3}h_{kk}\delta_{ij} + A\left(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij}\right) + 2\hat{k}_m\hat{k}_{(i}\epsilon_{j)mn}B_n + h^{\rm TT}_{ij}, \qquad \vec{k}\cdot\vec{B} = 0, \qquad \sum_i h^{\rm TT}_{ii} = 0, \qquad k^i h^{\rm TT}_{ij} = 0. \tag{23}$$

The first piece is just the trace. The scalar field A is proportional to the double-divergence of the trace-subtracted h_{ij}: $A = (3/2)(h_{ij} - \frac{1}{3}h_{kk}\delta_{ij})\hat{k}^i\hat{k}^j$. The transverse-vector field $\vec{B}$ is obtained by taking the gradient on one index and the curl on the other:

$$B_q = \hat{k}_i\hat{k}_p\epsilon_{jpq}h_{ij}, \tag{24}$$

or, in real space,

$$\vec{B} = \nabla^{-2}\left(\vec{\nabla} \times \vec{\nabla} \cdot \mathbf{h}\right). \tag{25}$$

Finally, the symmetric **transverse-trace-free tensor** $h_{ij}^{\rm TT}$ (which therefore only has 2 independent components for each $\vec{k}$) is obtained by using the **TT projection operator** twice: defining $P_{ij} \equiv \delta_{ij} - \hat{k}_i\hat{k}_j$, we have

$$h_{ij}^{\rm TT} = P_{im}P_{jn}h_{mn} - \frac{1}{2}P_{ij}(P_{mn}h_{mn}). \tag{26}$$

We can rewrite this in real space as a double-Laplace equation:

$$\nabla^2\nabla^2 h_{ij}^{\rm TT} = \left(\delta_{im}\nabla^2 - \partial_i\partial_m\right)\left(\delta_{jn}\nabla^2 - \partial_j\partial_n\right)h_{mn} - \frac{1}{2}(\delta_{ij}\nabla^2 - \partial_i\partial_j)\left(\nabla^2 h_{mm} - \partial_m\partial_n h_{mn}\right), \tag{27}$$

which uniquely defines the TT part of the metric (or of any tensor), given appropriate boundary conditions at spatial infinity. Note that throughout we have not worried about the placement of purely spatial indices (they are raised and lowered with $\eta_{ij} = \delta_{ij}$), and repeated indices are summed over, even if they are not up and down.

To recap, the 10 components of any symmetric, second-rank Lorentz-tensor field can be decomposed into 4 scalars [h_{00}, $h_{0i,i}$, h_{ii}, and $\partial_i\partial_j h_{ij}$, or linear combinations thereof], two transverse vectors [the curl of h_{0i} and the curl of $h_{ij,j}$], each with 2 independent components, and one transverse-trace-free tensor $h_{ij}^{\rm TT}$, also with 2 independent components.