General Relativity Fall 2018
Lecture 12: Linearized gravity, Part II

Yacine Ali-Haïmoud

A. Summary from previous lecture

We are considering nearly flat spacetimes with nearly globally Minkowski coordinates: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $|h_{\mu\nu}| \ll 1$. Such coordinates are not unique: provided we make small changes of coordinates, or gauge transformations $x^\mu \rightarrow x^\mu - \xi^\mu$, the metric perturbation remains small and changes as $h_{\mu\nu} \rightarrow h_{\mu\nu} + 2\xi_{(\mu,\nu)}$. We can see the metric perturbation as a Lorentz-tensor field on the flat Minkowski background. Since we will linearize the relevant equations, we may work in Fourier space: each Fourier mode satisfies an independent equation. We denote by $\vec{k}$ the wavenumber and by $\hat{k}$ its direction and k its norm.

We have decomposed the 10 independent components of the metric perturbation according to their transformation properties under spatial rotations: there are 4 independent "scalar" components, which can be taken, for instance, to be $h_{00}, \hat{k}^i h_{0i}, h_{ii}$, and $\hat{k}^i\hat{k}^j h_{ij}$ – or any 4 linearly independent combinations thereof. There are 2 independent transverse "vector" components, each with 2 independent components: $\epsilon^{ilm}\hat{k}_l h_{0m}$ and $\epsilon^{ilm}\hat{k}_l h_{mj}\hat{k}^j$ – these are proportional to the curl of h_{0i} and to the curl of the divergence of h_{ij}, and are divergenceless (transverse to the Fourier wavenumber $\vec{k}$). Finally, there is a "tensor" mode $h_{ij}^{\rm TT}$, which is the transverse-trace-free part of h_{ij}, and is obtained by double-projecting h_{ij} transverse to $\vec{k}$, and subtracting the trace. This piece has 2 independent components.

Before starting, let us write explicitly the gauge transformation equations in Fourier space: replace ∂_j by ik_j:

$$h_{00} \rightarrow h_{00} + 2\partial_0\xi_0, \qquad h_{0j} \rightarrow h_{0j} + \partial_0\xi_j + ik_j\xi_0, \qquad h_{jl} \rightarrow h_{jl} + 2ik_{(j}\xi_{l)}. \tag{1}$$

B. Gauge-invariant variables

While a gauge transformation in electromagnetism amounts to providing one scalar function, a gauge transformation in linearized GR amounts to providing 4 functions $\xi^0, \vec{\xi}$. These 4 functions can be decomposed in 2 scalars $\xi^0, \hat{k}_i\xi^i$ and a transverse vector $\epsilon_{ijk}\hat{k}^j\xi^k$. Therefore, we expect that **out of the 4 scalars components, only 2 linear combinations are gauge-invariant**. Similarly, **out of the 2 vector modes, only 1 linear combination is gauge-invariant**. Finally, since there is no way to make a TT mode out of scalars and vectors, we expect, and will show explicitly, that **the "tensor" mode is gauge-invariant**.

The two scalar gauge-invariant variables are not unique (any linear combination is also gauge-invariant). We'll see that these two make expressions particularly simple (for reference these are related to the gauge-invariant Bardeen potentials in cosmological perturbation theory):

$$\Psi \equiv \frac{1}{4}\left(\hat{k}_j\hat{k}_l h_{jl} - h_{jj}\right), \tag{2}$$

$$\Phi \equiv -\frac{1}{2}\left[h_{00} + \frac{2i\hat{k}_j}{k}\partial_0 h_{0j} - \frac{3}{2k^2}\partial_0^2\left(\hat{k}_j\hat{k}_l h_{jl} - \frac{1}{3}h_{jj}\right)\right]. \tag{3}$$

Exercise: check gauge-invariance explicitly and please report likely typos!

There is only one gauge-invariant vector mode, defined up to a normalization constant:

$$v^i \equiv \epsilon^{ilm}\hat{k}_l\left(h_{0m} + \frac{i}{k}\partial_0 h_{mj}\hat{k}_j\right) \tag{4}$$

Indeed, under a gauge transformation, the change in the parenthesis is a pure gradient, so its curl is zero.

Finally, since a gauge transformation cannot add a TT part to the metric: **the TT part of the metric perturbation is gauge-invariant**, much like the transverse-vector part of the vector potential is in electromagnetism. **Exercise:** show this explicitly by considering a gauge transformation and applying the TT operator.

So, to summarize, gauge freedom implies that there are only **6 physical degrees of freedom in the metric perturbation** (that we could tell right away just from counting the number of coordinates). For linearized GR, we can moreover explicilty identify these degrees of freedom and classify them as 2 scalar modes, 1 transverse vector mode, and 1 transverse-traceless "tensor" mode.

Note: it is always possible to set the metric to be flat and with vanishing first derivatives at any given point. So it is no surprise that the gauge-invariant variables are defined with at least two derivatives!

C. Linearized Einstein tensor

We now derive the explicit form of the Einstein tensor, so we can spell out the equations of motion of $h_{\mu\nu}$. This can be done by brute calculatory force: (*i*) compute the Christoffel symbols, (*ii*) compute the Riemann tensor (*iii*) contract it to get the Ricci tensor (*iv*) subtract the trace, all of this at linear order. You are encouraged to do so as an exercise (see also Carroll 7.2).

Instead, we will see here that we can in fact derive the linearized Einstein tensor, up to a single overall prefactor, from symmetry considerations.

We know that the Riemann, hence Einstein tensor is built of second derivatives of the metric. The Einstein tensor is a generally covariant tensor, and a fortiori, transforms like a tensor under Lorentz transformations. Our goal is therefore to write a 2-index Lorentz-covariant tensor out of the 4-index Lorentz-covariant tensor $\partial_a \partial_b h_{cd}$. To do so, we can contract any pair of indices (and properly symmetrize), and we may also contract 2 pairs of indices and multiply by the Lorentz metric $\eta_{\mu\nu}$. So the **most general form of the linearized Einstein tensor** (or of any rank (0,2) Lorentz-tensor built from the metric) is

$$G_{\mu\nu} = A\ \Box h_{\mu\nu} + 2B\ \partial^\alpha \partial_{(\mu} h_{\nu)\alpha} + C\ \partial_\mu \partial_\nu h + D\ \eta_{\mu\nu} \Box h + E\ \eta_{\mu\nu} \partial^\alpha \partial^\beta h_{\alpha\beta}, \tag{5}$$

where $h \equiv h_\mu{}^\mu$ is the trace of the metric perturbation, and all indices are lowered and raised with the flat metric. The coefficients A, B, C, D, E are just numbers, which must be the same regardless of $h_{\mu\nu}$.

Now we use the **Bianchi identity**, which to linear order takes the form $G_{\mu\nu}{}^{,\nu} = 0$:

$$0 = (A + B)\Box \partial^\nu h_{\mu\nu} + (C + D)\Box \partial_\mu h + (B + E)\partial_\mu \partial^\alpha \partial^\beta h_{\alpha\beta}. \tag{6}$$

This must hold for *any* metric perturbation h, regardless of whether it is actually a solution of EFE's or not: this is a purely geometric property of the Einstein tensor. This implies that $A + B = C + D = B + E = 0$. Hence we have reduced our Einstein tensor to

$$G_{\mu\nu} = A\ \left(\Box h_{\mu\nu} - 2\ \partial^\alpha \partial_{(\mu} h_{\nu)\alpha} + \eta_{\mu\nu} \partial^\alpha \partial^\beta h_{\alpha\beta}\right) + C\left(\partial_\mu \partial_\nu h - \eta_{\mu\nu} \Box h\right). \tag{7}$$

Finally, we use the fact that the components of the Einstein tensor are **gauge-invariant at linear order**. Under a gauge tranformation, we have

$$\begin{aligned} 0 = \Delta G_{\mu\nu} &= 2A\ \left(\Box \xi_{(\mu,\nu)} - \partial^\alpha \partial_\mu \xi_{(\nu,\alpha)} - \partial^\alpha \partial_\nu \xi_{(\mu,\alpha)} + \eta_{\mu\nu} \partial^\alpha \partial^\beta \xi_{\alpha,\beta}\right) + 2C\left(\partial_\mu \partial_\nu \xi^\alpha{}_{,\alpha} - \eta_{\mu\nu} \Box \xi^\alpha{}_{,\alpha}\right) \\ &= 2(A - C)\left(\eta_{\mu\nu} \Box - \partial_\mu \partial_\nu\right) \xi^\alpha{}_{,\alpha}. \end{aligned} \tag{8}$$

Again, that this must hold for any gauge transformation implies $A = C$. We therefore have obtained the **Einstein tensor, up to an overall normalization A**. To get the coefficient $A = -1/2$ we may pick a very simple metric, for instance assume only $h_{00} \neq 0$ and $\partial_t h_{00} = 0$ (the Newtonian limit). You will do this as an **Exercise**.

To conlcude, the linearized Einstein tensor is

$$G_{\mu\nu} = -\frac{1}{2}\ \left(\Box h_{\mu\nu} - 2\ \partial^\alpha \partial_{(\mu} h_{\nu)\alpha} + \eta_{\mu\nu} \partial^\alpha \partial^\beta h_{\alpha\beta} + \partial_\mu \partial_\nu h - \eta_{\mu\nu} \Box h\right) = 8\pi T_{\mu\nu} \tag{9}$$

D. Einstein field equations

We first note that we can also decompose $G_{\mu\nu}$ in scalars, vectors and tensors, and that each kind can only include the components of $h_{\mu\nu}$ of the same kind: one cannot make a vector mode out of a scalar, and vice-versa.

Let us now explicitly write the EFEs. The 00 equation can be rewritten in terms of our gauge-invariant variable Ψ,

$$\nabla^2 \Psi = 4\pi T_{00}. \tag{10}$$

This is just Poisson's equation! As in the case of electromagnetism, the (linearized) Bianchi identity $G_{0i,i} = G_{00,0}$, consistent with the conservation of the stress-energy tensor $T_{\mu\nu}{}^{,\nu} = 0$ (at linear order), implies that the divergence of the $0i$ equation is nothing but the time-derivative of the 00 equation, and therefore does not carry any additional information. The curl of the $0i$ equation can be written in terms of the gauge-invariant vector field as [**Exercise:** check and report any typos!]:

$$\nabla^2 v^i = 16\pi\epsilon^{ilm}\hat{k}_l T_{0m} \tag{11}$$

Out of the 6 purely spatial equations, 3 are redundant with the G_{0i} equations, again, from the Bianchi identites. We are left with three independent equations. The first one can be taken to be $G_{ij} - \frac{1}{3}\delta_{ij}G_{kk}$, which, upon taking the double gradient, gives us [please report typos!]

$$\nabla^4(\Phi - \Psi) = -12\pi\partial_i\partial_j\left(T_{ij} - \frac{1}{3}\delta_{ij}T_{kk}\right). \tag{12}$$

Finally, the transverse-trace-free part of G_{ij} gives us the following equation for the TT part of the metric perturbation:

$$\Box h_{ij}^{\rm TT} = -16\pi T_{ij}^{\rm TT}. \tag{13}$$

As expected, the EFEs only provide information about the 6 gauge-invariant, physical degrees of freedom. To fix all 10 components of $h_{\mu\nu}$ one needs to impose 4 additional and freely specifiable gauge (or coordinate) conditions.

E. Constraints and dynamics

Now, let us consider the character of these equations. The first three are **constraint equations**: they do not involve any time derivatives. It is easier to see this explicitly in the **transverse gauge**, which is the generalization of the Coulomb gauge in electromagnetism: this gauge is defined as

$$\partial_i h_{0i} = 0 = \partial_i\left(h_{ij} - \frac{1}{3}\delta_{ij}h_{kk}\right). \tag{14}$$

First, one needs to show explicitly that such a gauge choice is indeed possible, and it is: pick $\nabla^2\xi^i + \frac{1}{3}\partial_j\xi_{l,l} = -2\partial_j(h_{ij} - \frac{1}{3}\delta_{ij}h_{kk})$ and $\nabla^2\xi^0 = \partial_i h_{0i} + \partial_0\xi^i_{,i}$. In this gauge, we have $\Psi = -\frac{1}{6}h_{kk}$ and $\Phi = -\frac{1}{2}h_{00}$, and $v^i = \epsilon^{ilm}\hat{k}_l h_{0m}$, i.e. no time derivatives appear, and the equations for Φ, Ψ and v^i are clearly purely spatial, constraint equations.

Finally, the TT part is a truly dynamical equation for the gauge-invariant $h_{ij}^{\rm TT}$. Therefore, just like electromagnetism, **GR has two dynamical degrees of freedom**, the TT part of the metric. These are nothing but the famous **gravitational waves**, which propagate at the speed of light (they satisfy the wave equation in vaccum), and **can exist even in vacuum**, while all other components can be set to zero in vacuum by appropriate gauge choices.

While we have shown all this in linearized gravity, let's now mention how this carries over to non-linear GR. First, it remains true at the non-linear level that there are 6 physical degrees of freedom, due to the 4 coordinate degrees of freedom. However, one can no longer classify them into "scalars", "vectors" and "tensors" without a flat background. At the non-linear level, all these mix-up; for instance, the non-linear quantity $h_{ij}^{\rm TT}h_{ij}^{\rm TT}$ is a scalar.

Second, it remains true even in non-linear GR that the 0μ **Einstein field equations are constraint equations**: they do not contain any second-time derivative (in fact, the G_{00} equation contains no time derivative at all). This can be seen using the Bianchi identity (**Exercise**). It also remains true that the metric only has two dynamical degrees of freedom, though, again, one cannot explicitly identify them in non-linear gravity.

F. The harmonic gauge

We define the **trace-reversed metric perturbation**

$$\overline{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h. \tag{15}$$

The harmonic or Lorenz gauge is the generalization of the Lorenz gauge in electromagnetism, and consists of the 4 conditions

$$\partial^\mu\overline{h}_{\mu\nu} = 0 \tag{16}$$

It is tedious but straightforwad to show that the linearized Einstein field equations in this gauge take the particularly simple form

$$\Box\overline{h}_{\mu\nu} = -16\pi T_{\mu\nu}. \tag{17}$$

G. Geodesics and Newtonian approximation

Consider geodesics with 4-momentum p^α and affine parameter λ (s.t. $p = d/d\lambda$). We derived in homework 5 that the geodesic equation can be rewritten as

$$\frac{dp_\sigma}{d\lambda} = \frac{1}{2} p^\gamma p^\delta g_{\gamma\delta,\sigma}. \tag{18}$$

In particular, if the metric is independent of time (so that $\partial_{(0)}$ is a Killing vector field), then the *covariant* component of 4-momentum p_0 is conserved along geodesics. Let us understand what this means. Consider a massive particle. The 4-momentum has norm squared

$$m^2 = -g_{\mu\nu} p^\mu p^\nu = (p^0)^2 \left[1 - h_{00} - 2h_{0i} v^i - g_{ij} v^i v^j\right] \qquad v^i \equiv \frac{p^i}{p^0}. \tag{19}$$

The relation between p_0 and p^0 is

$$p_0 = g_{0\mu} p^\mu = -p^0 \left[1 - h_{00} - h_{0i} v^i\right], \tag{20}$$

so that, to linear order in perturbations, we get

$$|p_0| = \frac{m}{\sqrt{1 - g_{ij} v^i v^j}} (1 - \frac{1}{2} h_{00}). \tag{21}$$

If we moreover assume that $v \ll 1$, and recall that the Newtonian potential is $\phi = -\frac{1}{2} h_{00}$ (see Lecture 3) then we get, to lowest order,

$$|p_0| = m + \frac{1}{2} m \vec{v}^2 + m\phi. \tag{22}$$

Hence we see that the conservation of p_0 in the time-independent case is consistent with the conservation of Newtonian energy: rest mass + kinetic + potential.