General Relativity Fall 2018
Lecture 13: Far-field metric of a quasi-Newtonian source

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We consider sources that (i) weakly curve space-time, so that $|h_{\mu\nu}| \ll 1$ and (ii) are in slow-motion, so that, for any component of the stress-energy tensor, $|\partial_0 T_{\mu\nu}/\partial_i T_{\mu\nu}| \sim V \ll 1$, where V is the characteristic velocity of the sources.

Einstein's field equation implies $\partial^2 h_{\mu\nu} \sim T_{\mu\nu}$, so we can think of $T_{\mu\nu}$ as a linear-order quantity, and neglect quantities of order $h \times T$. In particular, at linear order, conservation of the stress energy tensor $\nabla^\mu T_{\mu\nu} = 0$ becomes $\partial^\mu T_{\mu\nu} = 0$. This implies $T_{ij}/T_{0i} \sim T_{0i}/T_{00} \sim V$.

To keep track of the order in the characteristic velocity V, we will write a bookkeeping parameter V^n in front of the appropriate terms. For instance, $\partial_0 T_{00} \sim V \partial_i T_{00}$, so we write $V\ \partial_0 T_{00}$. We will expand everything to $\mathcal{O}(V^3)$. When summing over repeated spatial indices, we are careless about putting them up or down, as we use the flat spatial metric δ_{ij} whenever dealing with perturbed quantities.

A. Preliminaries: mass, linear and angular momentum, moment of inertia, quadrupole moment

Before starting, let us define the following quantities, which will appear in the calculation:

$$M(t) \equiv \int d^3y\ T_{00}(t, \vec{y}), \qquad \vec{X}_{\rm cm}(t) \equiv \frac{1}{M}\int d^3y\ \vec{y}\ T_{00}(t,\vec{y}), \qquad \vec{P}_{\rm cm}(t) \equiv -\int d^3y\ T_{0i}(t,\vec{y}), \tag{1}$$

$$J^i \equiv \epsilon_{ijk}\int d^3y\ y^j T^{0k} = -\epsilon_{ijk}\int d^3y\ y_j T_{0k}. \tag{2}$$

These are, respectively, the total mass, center-of-mass position, linear momentum, and angular momentum of the source. Using the conservation of stress-energy tensor $\partial_0 T_{00} = \partial_i T_{0i}$ (to linear order in perturbations) and integrating by parts, we find that $\dot{M} = 0$ and $M\dot{\vec{X}}_{\rm cm} = \vec{P}_{\rm cm}$. We also find

$$\dot{P}^i_{\rm cm} = \int d^3y\ \partial_0 T^{0i} = -\int d^3y\ \partial_k T^{ki} = 0, \tag{3}$$

after integrating by parts over the finite source. Similarly, we have

$$\dot{J}_i = -\epsilon_{ijk}\int d^3y\ y_j \partial_0 T_{0k} = -\epsilon_{ijk}\int d^3y\ y_j \partial_l T_{lk} = \epsilon_{ijk}\int d^3y\ T_{jk} = 0. \tag{4}$$

Therefore, *at linear order in perturbations*, the total mass, linear momentum and angular momentum of the sources are conserved. This does not account for the loss of energy, momentum and angular momentum by gravitational-wave radiation, which is quadratic in metric perturbations. We will get back to this in a few lectures.

We also define the tensor of inertia I_{ij} and the quadrupole momentu Q_{ij} as follows:

$$I_{ij}(t) \equiv \int d^3y\ y_i y_j\ T_{00}(t,\vec{y}), \qquad Q_{ij} \equiv I_{ij} - \frac{1}{3} I_{kk}\delta_{ij}. \tag{5}$$

It will be useful to compute derivatives of the inertia tensor:

$$\dot{I}_{ij} = \int d^3y\ y_i y_j\ \partial_0 T_{00} = \int d^3y\ y_i y_j\ \partial_k T_{k0} = -\int d^3y\ (y_i T_{0j} + y_j T_{0i}) = -2\int d^3y\ y_{(i}T_{j)0} \tag{6}$$

$$\ddot{I}_{ij} = -2\int d^3y\ y_{(i}\partial_0 T_{j)0} = -\int d^3y\ (y_i \partial_k T_{kj} + y_j \partial_k T_{ki}) = 2\int d^3y\ T_{ij}. \tag{7}$$

Finally, we have the following equality:

$$\epsilon_{ilm} J^i = -\epsilon_{ilm}\epsilon_{ijk}\int d^3y\ y_j T_{0k} = -2\int d^3y\ y_{[l}T_{0m]}, \tag{8}$$

so that, combined with Eq. (6), we find

$$\int d^3y\ y_i T_{j0} = -\frac{1}{2}\left[\dot{I}_{ij} + \epsilon_{ijm} J^m\right]. \tag{9}$$

In what follows we boost the global coordinate system to a frame where $\vec{P}_{\rm cm} = \vec{0}$ (i.e. we make a global Lorentz transformation that enforces this condition).

B. Computation of $\overline{h}_{\mu\nu}$ in the harmonic gauge

In the harmonic gauge $\partial\overline{h}_{\mu\nu} = 0$, we saw that the linearized Einstein field equations become $\Box\overline{h}_{\mu\nu} = -16\pi T_{\mu\nu}$. This has the following integral solution

$$\overline{h}_{\mu\nu}(t,\vec{x}) = 4\int d^3y \frac{T_{\mu\nu}(t_{\rm ret},\vec{y})}{|\vec{x}-\vec{y}|}, \qquad t_{\rm ret} \equiv t - |\vec{x}-\vec{y}|. \tag{10}$$

We saw before that the integral of a general tensor over spacetime is meaningless. This integral should be interpreted as the integral of a Lorentz-tensor (i.e. a special-relativity tensor), over flat space.

Let us compute the solution far from the source, i.e. at distances $r = |\vec{x}| \gg L$. The retarded time can be expanded:

$$t_{\rm ret} = t - r + (\hat{x}\cdot\vec{y}) + \mathcal{O}(1/r), \tag{11}$$

so that

$$T_{\mu\nu}(t_{\rm ret}) = T_{\mu\nu}(t-r) + V\ (\hat{x}\cdot\vec{y})\partial_0 T_{\mu\nu}(t-r) + \frac{1}{2}\ V^2\ (\hat{x}\cdot\vec{y})^2\partial_0^2 T_{\mu\nu}(t-r) + \mathcal{O}(V/r, V^3)T_{\mu\nu}. \tag{12}$$

The first component of $\overline{h}_{\mu\nu}$ is then

$$\frac{1}{4}\overline{h}_{00} = \int d^3y\frac{T_{00}(t-r,\vec{y})}{|\vec{x}-\vec{y}|} + \frac{1}{r}\left[V\hat{x}\cdot\int d^3y\ \vec{y}\ \partial_0 T_{00} + \frac{1}{2}V^2\hat{x}^k\hat{x}^k\int d^3y\ y_k y_l\ \partial_0^2 T_{00} + \mathcal{O}(V/r, V^3)T_{00}\right], \tag{13}$$

where we only considered the $1/r$ term in the last two terms, consistently with our neglect of terms of order V/r.

The first term is just the opposite of the Newtonian potential at the retarded time:

$$\Phi_{\rm Newt}(t-r,\vec{x}) \equiv -\int d^3y\frac{T_{00}(t-r,\vec{y})}{|\vec{x}-\vec{y}|}. \tag{14}$$

The second term is proportional to $\vec{P}_{\rm cm}$, which we have set to zero by appropriate choice of the global Lorentz frame. We then find

$$\frac{1}{4}\overline{h}_{00}(t,\vec{x}) = -\Phi_{\rm Newt} + \frac{1}{2r}V^2\hat{x}^i\hat{x}^j\ddot{I}_{ij} + \mathcal{O}(V^3). \tag{15}$$

Let us now consider $\overline{h}_{0i}$. Since $T_{0i} \sim VT_{00}$, we only need to consider the first time derivative at the same order of approximation:

$$\frac{1}{4}\overline{h}_{0i} = V\int d^3y\frac{T_{0i}(t-r,\vec{y})}{|\vec{x}-\vec{y}|} + \frac{V^2}{r}\hat{x}^k\cdot\int d^3y\ y_k\ \partial_0 T_{0i} + \mathcal{O}(V^3). \tag{16}$$

The first term can also be expanded in L/r. The first term of this expansion is proprotional to $\vec{P}_{\rm cm}$, which we have set to zero. Therefore, to lowest order,

$$\frac{1}{4}\overline{h}_{0i} = \frac{V}{r^2}\hat{x}^k\int d^3y\ y_k T_{0i} + \frac{V^2}{r}\hat{x}^k\cdot\int d^3y\ y_k\ \partial_0 T_{0i}. \tag{17}$$

The first term can be re-expressed using Eq. (9). Using $\partial_0 T_{0i} = \partial_j T_{ji}$, we may integrate the last term by parts, and find it to be equal to

$$\int d^3y\ y_k\ \partial_0 T_{0i} = -\int d^3y T_{ik} = -\frac{1}{2}\ddot{I}_{ik}, \tag{18}$$

So we can rewrite $\overline{h}_{0i}$ as

$$\frac{1}{4}\overline{h}_{0i} = \frac{V}{2r^2}(\hat{x}\times\vec{J})^i - \frac{V}{2r^2}\hat{x}_k\dot{I}_{ki} - \frac{V^2}{2r}\hat{x}^k\ddot{I}_{ki}. \tag{19}$$

Finally, to lowest order,

$$\frac{1}{4}\overline{h}_{ij} = \frac{V^2}{2r}\ddot{I}_{ij}. \tag{20}$$

To summarize, we have obtained, up to terms of order V^3 (and now setting our bookkeeping parameter $V = 1$),

$$\overline{h}_{00} = -4\Phi_{\rm Newt} + \frac{2}{r}\hat{x}^i\hat{x}^j\ddot{I}_{ij}, \tag{21}$$

$$\overline{h}_{0i} = \frac{2}{r^2}(\hat{x}\times\vec{J})^i - \frac{2}{r^2}\hat{x}_k\dot{I}_{ki} - \frac{2}{r}\hat{x}^k\ddot{I}_{ki}, \tag{22}$$

$$\overline{h}_{ij} = \frac{2}{r}\ddot{I}_{ij}. \tag{23}$$

C. Complutation of $h_{\mu\nu}$ and gauge transformation

Let us now compute $h_{\mu\nu} = \overline{h}_{\mu\nu} - \frac{1}{2}\overline{h}\eta_{\mu\nu}$. The trace of $\overline{h}_{\mu\nu}$ is

$$\overline{h} = -\overline{h}_{00} + \overline{h}_{ii} = 4\Phi_{\rm Newt} + \frac{2}{r}\left(\ddot{I}_{kk} - \hat{x}^k\hat{x}^l\ddot{I}_{kl}\right), \tag{24}$$

so that we find

$$h_{00} = -2\Phi_{\rm Newt} + \frac{1}{r}\left(\ddot{I}_{kk} + \hat{x}^k\hat{x}^l\ddot{I}_{kl}\right), \tag{25}$$

$$h_{0i} = \frac{2}{r^2}(\hat{x}\times\vec{J})^i - \frac{2}{r^2}\hat{x}_k\dot{I}_{ki} - \frac{2}{r}\hat{x}^k\ddot{I}_{ki}, \tag{26}$$

$$h_{ij} = -2\Phi_{\rm Newt}\delta_{ij} + \frac{1}{r}\left[2\ddot{I}_{ij} - \left(\ddot{I}_{kk} - \hat{x}^k\hat{x}^l\ddot{I}_{kl}\right)\delta_{ij}\right]. \tag{27}$$

Recall that the right-hand-side is to be evaluated at $t - r$.

These expressions have been obtained in the harmonic gauge, but nothing prevents us now from going to another gauge where these expressions might be simpler. We now change gauges in order to eliminate all the inertia tensor terms from h_{00} and h_{0i}.

Recalling that $h_{\mu\nu} \to h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}$, we first eliminate the second term in h_{00} by chosing

$$\xi_0 = -\frac{1}{2r}\left(\dot{I}_{kk} + \hat{x}^k\hat{x}^l\dot{I}_{kl}\right)_{t-r}, \qquad \xi_i = 0, \tag{28}$$

so that we obtain $h_{00} = -2\Phi_{\rm Newt}$, up to corrections of order V^3. This gauge transformation also changes $h_{0i} \to h_{0i} + \partial_i\xi_0$, which gives us, after simplification, and remembering that ξ_0 also depends on r through th retarded time $t - r$,

$$h_{0i} = \frac{2}{r^2}(\hat{x}\times\vec{J})^i + \frac{1}{2r}\left(\ddot{I}_{kk} + \hat{x}^k\hat{x}^l\ddot{I}_{kl}\right)\hat{x}^i - \frac{2}{r}\hat{x}^k\ddot{I}_{ki} + \kappa_{kl}\frac{\dot{I}_{kl}}{r^2}, \tag{29}$$

where we neet not get the exact expression of κ_{kl} for the rest of the argument.

We now seek to cancel all but the first term in h_{0i}: we change gauges with

$$\xi_0 = 0, \qquad \xi_i = \frac{2}{r}\hat{x}^k\dot{I}_{ki} - \frac{1}{2r}\left(\dot{I}_{kk} + \hat{x}^k\hat{x}^l\dot{I}_{kl}\right)\hat{x}^i - \kappa_{kl}\frac{I_{kl}}{r^2}. \tag{30}$$

This transformation gives us $h_{0i} = 2/r^2(\hat{x}\times\vec{J})^i$, up to corrections of order V^3. It also changes $h_{ij} \to h_{ij} + \xi_{i,j} + \xi_{j,i}$. At large distances, the dominant, $1/r$ term is

$$\xi_{i,j} \approx -\dot{\xi}_i\hat{x}^j = \frac{1}{r}\left(\ddot{I}_{kk} + \hat{x}^k\hat{x}^l\ddot{I}_{kl}\right)\hat{x}^i\hat{x}^j - \frac{2}{r}\hat{x}^k\ddot{I}_{ki}\hat{x}^j. \tag{31}$$

Therefore, we find

$$h_{ij} = -2\Phi_{\rm Newt}\delta_{ij} + \frac{1}{r}\left[2\ddot{I}_{ij} - \left(\ddot{I}_{kk} - \hat{x}^k\hat{x}^l\ddot{I}_{kl}\right)\delta_{ij} + \left(\ddot{I}_{kk} + \hat{x}^k\hat{x}^l\ddot{I}_{kl}\right)\hat{x}^i\hat{x}^j - 2\hat{x}^k\ddot{I}_{ki}\hat{x}^j - 2\hat{x}^k\ddot{I}_{kj}\hat{x}^i\right] + \mathcal{O}(1/r^2). \tag{32}$$

This can be rewritten as

$$h_{ij} = -2\Phi_{\rm Newt}\delta_{ij} + \left(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right)\frac{2}{r}\ddot{I}_{kl}, \qquad P_{ij} \equiv \delta_{ij} - \hat{x}_i\hat{x}_j. \tag{33}$$

The projector in parenthesis is just the transverse-trace-free (TT) projector, at large distances. Indeed, we had defined it as a differential operator through $P_{ij} = \nabla^{-2}\left(\nabla^2\delta_{ij} - \partial_i\partial_j\right)$. Applying this to a function of the form $f(t-r)/r$, we see that the dominant term at large distances comes form the derivative of the retarded time, and is just $\delta_{ij} - \hat{x}^i\hat{x}^j$.

Since the TT part of $(I_{kk}/r)\delta_{ij}$ vanishes, we may equivalently write this in terms of the TT part of the quadrupole moment. To summarize, we got

$$h_{ij} = -2\Phi_{\rm Newt}\delta_{ij} + \left[\frac{2}{r}\ddot{Q}_{ij}\right]^{\rm TT}. \tag{34}$$

The last part, which is the gauge-invariant TT part of the metric, are the **gravitational waves**.

To summarize, we found that far from a quasi-Newtonian source, the metric can be cast in the form

$$\boxed{ds^2 = -(1+2\Phi_{\rm Newt})dt^2 + \frac{4}{r^2}(\hat{x}\times\vec{J})\cdot d\vec{x}dt + (1-2\Phi_{\rm Newt})\delta_{ij} + h^{\rm TT}_{ij}dx^i dx^j, \qquad h^{\rm TT}_{ij} = \left[\frac{2}{r}\ddot{Q}_{ij}\right]^{\rm TT}.} \tag{35}$$

The Newtonian potential itself can be expanded in the usual multipole expansion at large distance from the source: setting the origin of coordinates at the center of mass, so that $\vec{X}_{\rm cm} = 0$, we get

$$\Phi_{\rm Newt} = -\frac{M}{r} - \frac{3Q_{ij}}{2r^3}\hat{x}^i\hat{x}^j + \mathcal{O}(1/r^4). \tag{36}$$