General Relativity Fall 2018
Lecture 15: Gravitational lensing and Shapiro time delay

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GRAVITATIONAL LENSING

Propagation of electromagnetic waves in curved spacetime in the geometric optics limit

Here we explain why light rays propagate along null geodesics, starting from Maxwell's equations. This argument applies in the geometric optics limit, but in arbitrarily curved spacetimes.

In the Lorenz gauge $\nabla_\alpha A^\alpha = 0$, Maxwell's equations in vaccum become (see HW 4)

$$\nabla_\beta \nabla^\beta A^\alpha - R^\alpha{}_\gamma A^\gamma = 0. \tag{1}$$

These are linear equations for the vector potential A_α. We can first solve them for a complex vector potential $\mathcal{A}_\alpha = a_\alpha \mathrm{e}^{i\theta}$, such that $A_\alpha = \mathrm{Re}(\mathcal{A}_\alpha)$, and then take the real part. We define the wavevector $k_\alpha \equiv \nabla_\alpha \theta$, i.e. k is the one-form $d\theta$. It is orthogonal to surfaces of constant phase θ. We call **light rays** the curves orthogonal to these surfaces, with tangent vector k.

Let us rewrite Maxwell's equations explicitly:

$$0 = \nabla^\beta \left[\mathrm{e}^{i\theta}(\nabla_\beta a^\alpha + i k_\beta a^\alpha)\right] - \mathrm{e}^{i\theta} R^\alpha{}_\gamma a^\gamma. \tag{2}$$

Expanding and dividing by $\mathrm{e}^{i\theta}$, we arrive at

$$0 = \nabla^\beta \nabla_\beta a^\alpha + i\left(a^\alpha \nabla^\beta k_\beta + 2k^\beta \nabla_\beta a^\alpha\right) - k^\beta k_\beta a_\alpha - R^\alpha{}_\gamma a^\gamma. \tag{3}$$

We now assume that we are in the geometric optics limit, i.e. that the phase varies on a lengthscale λ much smaller than the characteristic radius of curvature of spacetime, $\mathcal{R} \sim \mathrm{Riemann}^{-1/2}$, and than the characteristic lengthscale over which the amplitude a_α varies. In other words, we have moved all the fast variations of the vector potential to the phase. This is the **geometric optics limit**.

In this limit, the Lorenz gauge implies $k_\mu a^\mu = 0$, to lowest order.

The dominant term in Maxwell's equation ($\propto k^2$) implies $k^\beta k_\beta = 0$, i.e. the wavenumber is a null vector.

Taking the gradient of this equality, we find

$$0 = \nabla_\mu(k_\nu k^\nu) = 2k^\nu \nabla_\mu k_\nu = 2k^\nu \nabla_\mu \nabla_\nu \theta = 2k^\nu \nabla_\nu \nabla_\mu \theta = 2k^\nu \nabla_\nu k_\mu, \tag{4}$$

where we used the fact that the covariant derivative is torsion-free.

Therefore, light rays, which are the curves orthogonal to the surfaces of constant phase, have null tangent vectors, which are parallel-transported along themselves. In other words, light rays are null geodesics. They are moroever transverse to the vector potential.

Liouville's theorem

Let us pick a LICS $\{x^\mu\}$. Such a coordinate system is determined up to a Lorentz transformation. Our first goal is to show that $dx^1 dx^2 dx^3 dp^1 dp^2 dp^3$ is Lorentz-invariant, so it is a well-defined phase-space volume element.

The 4-volume element $dx^0 dx^1 dx^2 dx^3$ is Lorentz-invariant, and so is the proper time τ, so that, dividing by τ, we see that $p^0 dx^1 dx^2 dx^3$ is Lorentz-invariant. The 4-dimensional momentum volume element $dp^0 dp^1 dp^2 dp^3$ is also Lorentz invariant, and so is

$$dp^1 dp^2 dp^3 \int dp^0 \delta_{\mathrm{D}}\left[\frac{1}{2}(\eta_{\mu\nu} p^\mu p^\nu + m^2)\right] = \frac{dp^1 dp^2 dp^3}{p^0}. \tag{5}$$

This concludes our proof. We can therefore unambiguously define the 6-dimensional phase-space volume element, and the *phase-space density*

$$\mathcal{N} \equiv \frac{dN_{\mathrm{particles}}}{dx^1 dx^2 x^3 dp^1 dp^2 dp^3}, \tag{6}$$

in a LICS, even if such a frame is defined only up to a Lorentz transformation.

Suppose we have N particles in some initial phase-space volume centered around momentum $\vec{p}$, that are not subject to any non-gravitational force. In a LICS, their momenta are constant, and their positions evolve according to $dx^i/dt = p^i/m$. After some time Δt, the positions are therefore $x'^i = x^i + \Delta t \times p^i/m$. The 6-dimensional volume at $t + \Delta t$ is the Jacobian of the $(x, p) \rightarrow (x', p')$ transformation, which is just unity. So the 6-dimensional phase-space volume enclosing the particles is conserved. As a consequence, **the phase-space density is conserved along particle trajectories (Liouville's theorem)**:

$$\frac{d\mathcal{N}}{dt}\Big|_{\rm traj} = 0. \tag{7}$$

Application in asymptotically-flat spacetime: flux $\propto$ solid angle subtended by image

We now rewrite explicitly the phase-space density in asymptotically flat regions:

$$\mathcal{N} = \frac{dN_{\rm part}}{p^2 dp d\hat{p} \ d\text{Area} \ dt}. \tag{8}$$

where we replaced the component of $\vec{x}$ along the "central" momentum $\vec{p}$ by dt, and the Area is perpendicular to the central momentum. The quantity $I \equiv p^3 \mathcal{N}$ is the *specific intensity*. Liouville's theorem implies that I/p^3 is constant. In the case of gravitational lensing by a stationary mass, we saw that $p \equiv |\vec{p}|$ is constant, which implies that the specific intensity itself is constant along photon trajectories.

The observed flux is the integral of the specific intensity over the solid angle subtended by the source. Assuming a source with uniform specific intensity, we therefore find that the observed flux is

$$F_{\rm obs} = I_{\rm source} \times \Delta\Omega_{\rm image}. \tag{9}$$

Therefore, lensing leads to a *magnification* of the observed flux, by the ratio of the solid angles subtended by the image and the source:

$$\mu \equiv \text{magnification} = \left| \det \left(\frac{\partial \hat{\Omega}_{\rm image}}{\partial \hat{\Omega}_{\rm source}} \right) \right|. \tag{10}$$

Application to gravitational lensing

We now use our results from previous lecture for geodesic deviation, with the stationary far-field metric $ds^2 = -(1 - 2M/r)dt^2 + (1 + 2M/r)d\vec{x}^2$.

Refer to Figure 1 for the definition of angles and distances in the gravitational lensing problem. We make the thin lens-plane approximation, and assume that light deflection all happens in a single plane. In the small-angle approximation, the transverse distance $X \approx \alpha d_{\rm S} \approx |\Delta\hat{p}| d_{\rm LS} = 4M d_{\rm LS}/b$. We also have $b = \theta d_{\rm L}$, so that $\alpha = 4M d_{\rm LS}/(d_{\rm L} d_{\rm S} \theta)$. We may then relate the observed position of the image, θ, to the true (and unobserved) position of the source, $\beta = \theta - \alpha$:

$$\beta = \theta - \frac{\theta_{\rm E}^2}{\theta}, \quad \theta_{\rm E} \equiv \sqrt{\frac{4M d_{\rm LS}}{d_{\rm S} d_{\rm L}}}. \tag{11}$$

The angular scale $\theta_{\rm E}$ is the **Einstein radius**: it is such if a source lies just behind the lens ($\beta = 0$), its image is a ring at $|\theta| = \theta_{\rm E}$, called an **Einstein ring**.

In general, given a true source position β, there are two images

$$\theta_{\pm} = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_{\rm E}^2}\right). \tag{12}$$

One of the two images is inside the Einstein ring, and the other outside. As $\beta/\theta_{\rm E} \rightarrow 0$, they converge to $\theta_+ = \theta_- = \theta_{\rm E}$. As $\beta/\theta_{\rm E} \rightarrow \infty$, we get $\theta_+ \approx \beta$, and $\theta_- \approx -\theta_{\rm E}^2/\beta \rightarrow 0$.

We can now compute the magnification from Eq. (10), for each image, using spherical polar coordinates with polar axis along the observer-lens axis. Lensing does not change the polar angle φ, and in the the small angle approximation, $\sin\theta \approx \theta$, so we get

$$\mu_\pm = \left|\frac{\theta_\pm d\theta_\pm}{\beta d\beta}\right| = \frac{\theta_\pm^2}{\beta\sqrt{\beta^2 + 4\theta_\mathrm{E}^2}}. \tag{13}$$

We then find the total magnification,

$$\mu_\mathrm{tot} = \mu_+ + \mu_- = \frac{\beta^2 + 2\theta_\mathrm{E}^2}{\beta\sqrt{\beta^2 + 4\theta_\mathrm{E}^2}} > 1. \tag{14}$$

Suppose, for instance, that the lens passes in front of the line of sight at constant velocity: $\beta^2 = \beta_\mathrm{min}^2 + (\dot{\beta}t)^2$. We show the resulting magnification as a function of time in Fig. 2. Large magnifications can be achieved if the source passes within the Einstein radius of the lens. This effect is used to look for compact halo objects in the Galactic halo.

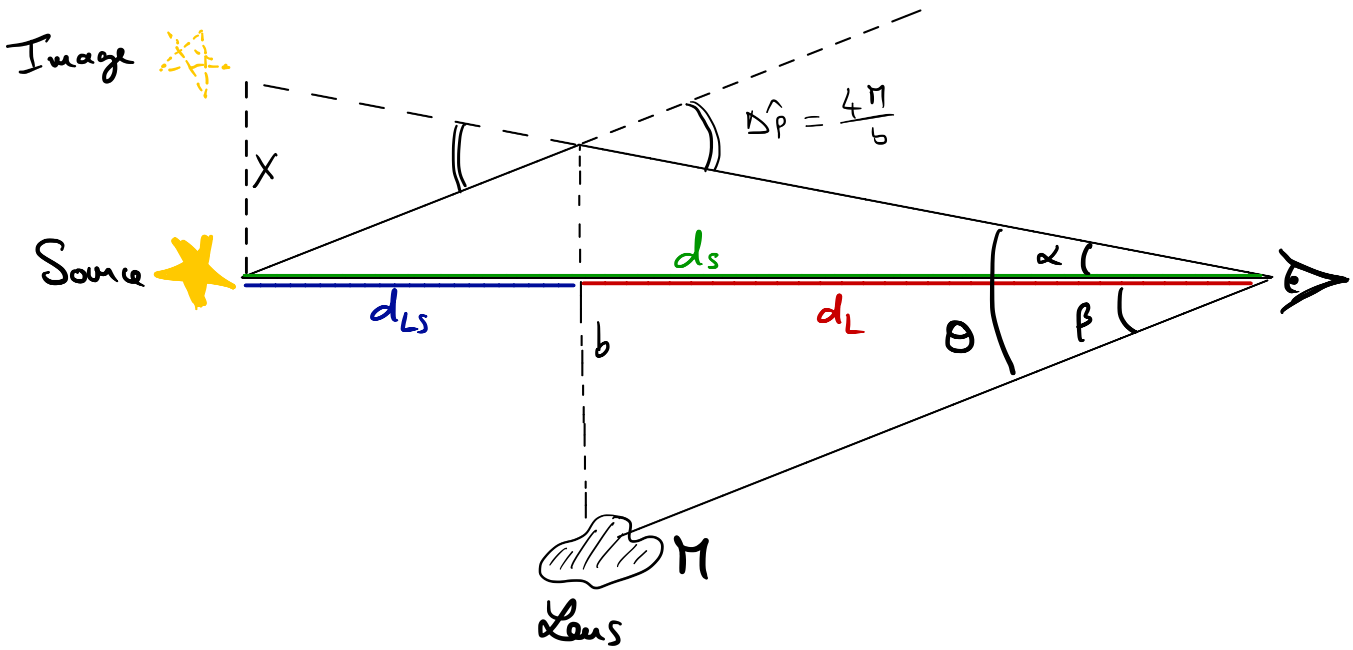

FIG. 1. Geometry of the gravitational lensing problem

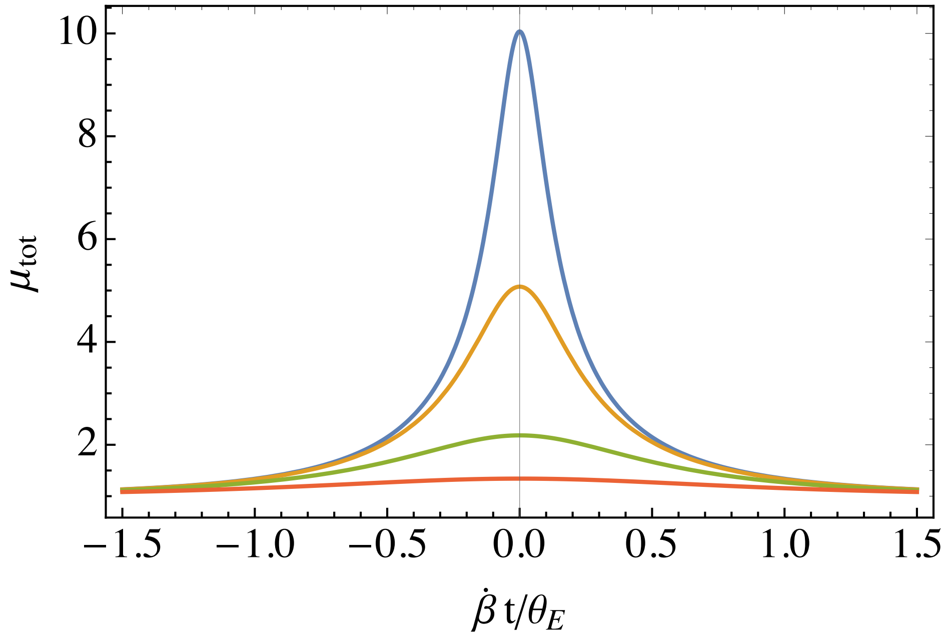

FIG. 2. Magnification by gravitational lensing for $\beta_{\rm min}/\theta_{\rm E} = 0.1, 0.2, 0.5$ and 1, from top to bottom.

SHAPIRO TIME DELAY

We had seen that $|\vec{p}| \equiv (\delta_{ij} p^i p^j)^{1/2} = (1+2\Phi)p^0 = -p_0$ is constant due to the stationarity of spacetime. Consider two points with coordinate separation $|\Delta\vec{x}| = d$. The coordinate time for a light signal to travel between them is

$$\Delta t = \int_{\rm traj} d\lambda \frac{dx^0}{d\lambda} = \int_{\rm traj} d\lambda\ p^0 = \int d\lambda \frac{|\vec{p}|}{1+2\Phi} = \int_{\rm traj} d\ell\ (1-2\Phi) = \ell_{\rm traj} - 2\int_{\rm traj} d\ell\ \Phi. \tag{15}$$

The distance along the trajectory, $\ell_{\rm traj}$, is slightly larger than the unperturbed separation d. This difference is a *geometric* time delay. We can estimate this by approximating photon trajectories as straight lines, as shown in the figure. We saw that the deflection angle is $|\Delta\hat{p}| = 4M/b$. The change in impact parameter is therefore $\Delta b \approx (d/2) \times 2M/b \approx dM/b$. The distance travelled is then $\ell_{\rm traj} \approx 2\sqrt{(d/2)^2 + \Delta b^2} \approx d + 2d(M/b)^2$. The geometric time delay is therefore of order $d(M/b)^2$.

In addition, the second term $2\int d\lambda \Phi$ is the *Shapiro time delay*. Let us estimate it by computing the integral along the unperturbed trajectory. Assume the source is at coordinate z_i and the receiver at coordinate z_f, we then have

$$\Delta t_{\rm Shapiro} = -2\int d\lambda\ \Phi \approx 2M \int_{z_i}^{z_f} \frac{dz}{\sqrt{b^2+z^2}} = 2M\left[\operatorname{arcsinh}(z_f/b) - \operatorname{arcsinh}(z_i/b)\right] \approx 2M \log\left(4\frac{|z_i| z_f}{b^2}\right), \tag{16}$$

where we assumed $b \ll |z_i|, z_f$.

The Shapiro time delay is considered a classic test of GR. It was measured by bouncing off radio signals from the surface of solar system planets (Mercury, Mars, Venus). In this case, $M = M_\odot \sim 10^{-5}$ sec $\sim 10^5$ cm, and characteristic distances are $d \sim$ AU $\sim 10^{13}$ cm, and, at closest, $b \sim R_\odot \sim 10^{11}$ cm, so that the geometric delay is of order 10^{-9} seconds, and the Shapiro delay is of order $\sim 10^{-4}$ seconds.