General Relativity Fall 2018
Lecture 16: Gravitational waves: polarizations, generation by a circular binary

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We now focus on the gauge-invariant transverse-trace free part of metric perturbations, $h_{ij}^{\rm TT}$, i.e. the gravitational waves (GWs). We neglect the effect of other metric perturbations, so for all intents and purposes, we assume that $ds^2 = -dt^2 + (\delta_{ij} + h_{ij}^{\rm TT})dx^i dx^j$.

In the next few lectures, we will discuss the effect of GWs on point masses, the polarization of GWs, their generation, and the energy and angular momentum they carry.

Polarizations of GWs

In vacuum, the Einstein field equations are $\Box h_{ij}^{\rm TT} = 0$. We may decompose $h_{ij}^{\rm TT}$ in Fourier modes:

$$h_{ij}^{\rm TT}(t, \vec{x}) = {\rm Re}\left[\int \frac{d\omega}{2\pi}\frac{d^3k}{(2\pi)^3} A_{ij}(\omega, \vec{k}) {\rm e}^{i(\omega t - \vec{k}\cdot\vec{x})}\right], \tag{1}$$

where A_{ij} is a complex symmetric, trace-free matrix. The vacuum propagation equation $\Box h_{ij}^{\rm TT} = 0$ implies $\omega^2 = \vec{k}^2$. The transverse condition $\partial_i h_{ij}^{\rm TT} = 0$ implies $k^i A_{ij} = 0$.

Consider a specific Fourier mode $\vec{k}$. Align the coordinate axes so that $\vec{k} \propto \hat{e}_3$, and so that $A_{i3} = 0$. Redefine the origin of time so that $A_{11} = -A_{22}$ is real. Write $A_{12} = h_\times + i h_C$, and define $A_{11} = h_+ + h_C$. We then have

$$\begin{aligned} h_{ij}^{\rm TT}(t, \vec{x}) &= {\rm Re}\left[\begin{pmatrix} h_+ + h_C & h_\times - i h_C \\ h_\times - i h_C & -h_+ - h_C \end{pmatrix} {\rm e}^{i(\omega t - \vec{k}\cdot\vec{x})}\right] \\ &= \begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix} \cos(\omega t - \vec{k}\cdot\vec{x}) + h_C \begin{pmatrix} \cos(\omega t - \vec{k}\cdot\vec{x}) & -\sin(\omega t - \vec{k}\cdot\vec{x}) \\ -\sin(\omega t - \vec{k}\cdot\vec{x}) & -\cos(\omega t - \vec{k}\cdot\vec{x}) \end{pmatrix}. \end{aligned} \tag{2}$$

The first component is a **linearly-polarized** GW, and the second component is **circularly polarized**. We will shortly understand why.

Transformation of $h_{ij}^{\rm TT}$ under rotations

Suppose in some coordinate system we have $h_\times = h_C = 0$, i.e.

$$h_{ij}^{\rm TT} = h_L \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

Let compute the components of $h_{ij}^{\rm TT}$ is a coordinate system rotated by φ:

$$\begin{aligned} h_{ij}^{\rm TT} &= h_L \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} = h_L \begin{pmatrix} \cos^2\varphi - \sin^2\varphi & 2\sin\varphi\cos\varphi \\ 2\sin\varphi\cos\varphi & \sin^2\varphi - \cos^2\varphi \end{pmatrix} \\ &= h_L \begin{pmatrix} \cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & -\cos 2\varphi \end{pmatrix}. \end{aligned} \tag{4}$$

Therefore we see that the components of $h_{ij}^{\rm TT}$ go back to themselves after a rotation by an angle π (the transverse-trace-free tensor $h_{ij}^{\rm TT}$ is said to have helicity 2). Also, we can identify $h_+ = h_L \cos(2\varphi)$ and $h_\times = h_L \sin(2\varphi)$, i.e. in an arbitrary basis, the linear polarization piece is just the result of rotating the diagonal form (3) by $\varphi = \frac{1}{2}\arctan(h_\times/h_+)$. So we can equivalently describe the linear polarization piece by the components $h_+, h_\times$ in a given, fixed coordinate system, or by the orientation of its eigenvectors (i.e. the angle φ with respect to the coordinate system in which it is diagonal) and $h_{\rm L} \equiv \sqrt{h_+^2 + h_\times^2}$.

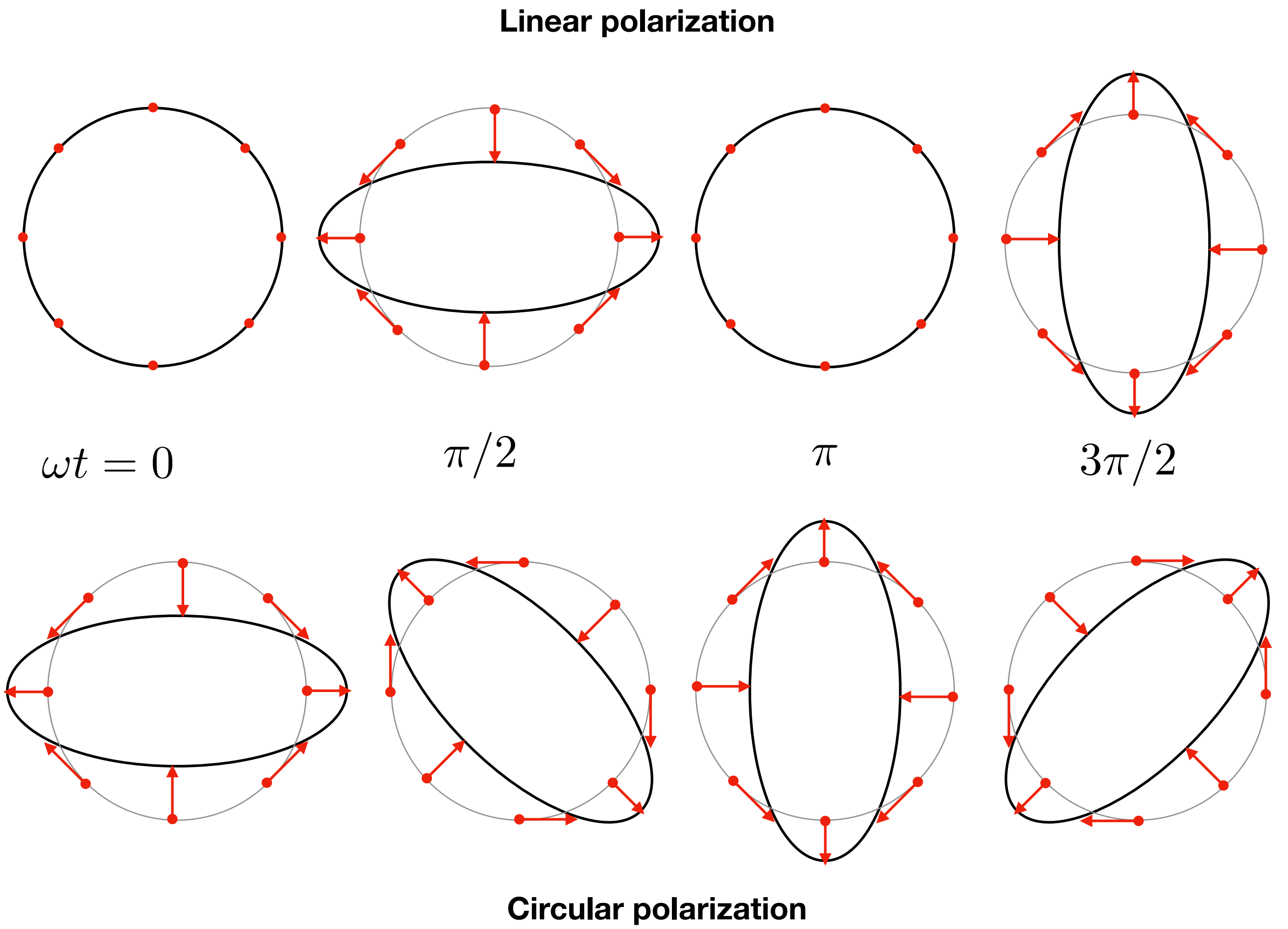

FIG. 1. Relative displacements of a circle of free-falling test masses under the influence of GWs. If the circle has radius R, the red arrows have length $(h_L/2)R$.

Effect on point masses

We derived in lecture 9 that the acceleration of separation of nearby geodesics is

$$\frac{d^2\Delta x^i}{dt^2} = -\Delta x^j R^i_{0j0}, \tag{5}$$

where here we assumed non-relativistic particles to $d\tau \approx dt$. The relevant components of the Riemann tensor can be easily computed and are just

$$R_{i0j0} = -\frac{1}{2}\partial_t^2 h_{ij}^{\rm TT}. \tag{6}$$

Suppose that $d(\Delta x^i)/dt = 0$ initially. Integrating this equation, we may replace $\Delta x^j = \Delta x^j(0)$ in the right-hand-side (to linear order in perturbations), and arrive at

$$\Delta x^i(t) = \Delta x^i(0) + \frac{1}{2}h_{ij}^{\rm TT}(t)\Delta x^j(0). \tag{7}$$

We see that separations are change by a fractional amount of order the **strain**, i.e. the amplitude of $h_{ij}^{\rm TT}$. The effect of linear or circular polarization on the coordinate separations of a ring of test masses is shown in Fig. 1.

GWs generated by a circular binary

We derived in Lecture 13 (asuming that the metric is quasi-Minkowski everywhere, including inside the sources), that the TT part of the metric is sourced by the second derivative of the moment of inertia tensor (or equivalently, of its trace-free part, the quadrupole moment):

$$h_{ij}^{\rm TT} = 2\left(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right)\frac{\ddot{I}_{kl}}{r}, \qquad I_{kl} \equiv \int d^3y\rho(y)y_k y_l. \tag{8}$$

Let us consider a binary with masses M_1, M_2, and total mas M. The separations from the center of mass as $\vec{R}_1 = -(M_2/M)\vec{R}$ and $\vec{R}_2 = (M_1/M)\vec{R}$, where $\vec{R} \equiv \vec{R}_2 - \vec{R}_1$ is the separation vector. The inertia tensor is therefore

$$I^{ij} = M_1 r_1^i r_1^j + M_2 r_2^i r_2^j = \frac{M_1 M_2}{M} R^i R^j. \tag{9}$$

Let us specialize to a circular orbit with semi-major axis a, so that

$$\vec{R} = a\left(\cos(\Omega t)\hat{u} + \sin(\Omega t)\hat{v}\right), \tag{10}$$

where $\Omega = M^{1/2}/a^{3/2}$ is the orbital angular frequency, and $\hat{u}$ and $\hat{v}$ unit vectors in the orbital plane. The tensor of inertia $\mathbf{I}$ is then

$$\begin{aligned}\mathbf{I} &= \frac{M_1M_2}{M}a^2\left[\cos^2(\Omega t)\hat{u}\otimes\hat{u} + \cos(\Omega t)\sin(\Omega t)(\hat{u}\otimes\hat{v} + \hat{v}\otimes\hat{u}) + \sin^2(\Omega t)\hat{v}\otimes\hat{v}\right] \\ &= \frac{M_1M_2}{2M}a^2\left[\cos(2\Omega t)(\hat{u}\otimes\hat{u} - \hat{v}\otimes\hat{v}) + \sin(2\Omega t)(\hat{u}\otimes\hat{v} + \hat{v}\otimes\hat{u})\right] + \frac{M_1M_2}{2M}a^2(\hat{u}\otimes\hat{u} + \hat{v}\otimes\hat{v}).\end{aligned} \tag{11}$$

Taking two time derivatives, we arrive at

$$\ddot{\mathbf{I}} = \frac{2M_1M_2}{M}(\Omega a)^2\left[\cos(2\Omega t)(\hat{v}\otimes\hat{v} - \hat{u}\otimes\hat{u}) - \sin(2\Omega t)(\hat{u}\otimes\hat{v} + \hat{v}\otimes\hat{u})\right]. \tag{12}$$

We see that the matrix of inertia, hence gravitational waves, oscillate at *twice* the orbital frequency. This would no longer be the case if the orbit were eccentric: in this case the inertia tensor would have components at all multiples (including 1) of the orbital frequency.

Let us now project $\ddot{I}_{ij}(t-r)/r$ with the traceless-transverse projector

$$\mathcal{P}_{ijkl}^{\rm TT} \equiv P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}. \tag{13}$$

We saw that at large distances r, $P_{ij} \approx \delta_{ij} - \hat{r}_i\hat{r}_j$.

Suppose we view the binary with some angle Θ to its angular momentum $\vec{L}$, as shown in Fig. 2. Redefine the origin of time so the vector $\hat{u}$ lies in the same plane as $\vec{L}$ and $\vec{r}$, and the vector $\hat{v}$ is orthogonal to it. Define the unit vector $\hat{w}$ to be the projection of $\hat{u}$ perpendicular to $\hat{r}$, normalized to unity:

$$\hat{w} \equiv \frac{\hat{u} - (\hat{u}\cdot\hat{r})\hat{r}}{||\hat{u} - (\hat{u}\cdot\hat{r})\hat{r}||} = \frac{\hat{u} - (\hat{u}\cdot\hat{r})\hat{r}}{|\cos\Theta|}. \tag{14}$$

We then have

$$P\hat{u} = |\cos\Theta|\ \hat{w}, \quad P\hat{v} = \hat{v}, \tag{15}$$

so that

$$\mathcal{P}^{\rm TT}(\hat{v}\otimes\hat{v}) = \frac{1}{2}(\hat{v}\otimes\hat{v} - \hat{w}\otimes\hat{w}), \tag{16}$$

$$\mathcal{P}^{\rm TT}(\hat{u}\otimes\hat{u}) = \frac{\cos^2\Theta}{2}(\hat{w}\otimes\hat{w} - \hat{v}\otimes\hat{v}), \tag{17}$$

$$\mathcal{P}^{\rm TT}(\hat{u}\otimes\hat{v}) = |\cos\Theta|(\hat{w}\otimes\hat{v}). \tag{18}$$

So we get

$$\mathbf{h}^{\rm TT} = \frac{2M_1M_2}{Mr}(\Omega a)^2\left[\cos(2\Omega t)(1+\cos^2\Theta)(\hat{v}\otimes\hat{v} - \hat{w}\otimes\hat{w}) - 2\sin(2\Omega t)|\cos\Theta|(\hat{w}\otimes\hat{v} + \hat{v}\otimes\hat{w})\right]. \tag{19}$$

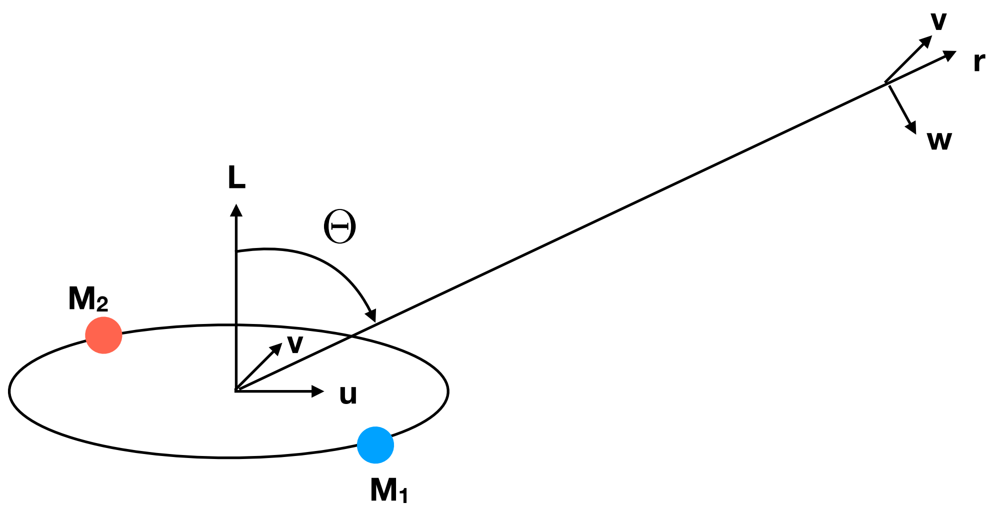

FIG. 2. Geometry of the problem of a circular binary radiating GWs.

Subtracting $\cos(2\Omega t)2|\cos\Theta|(\hat{v}\otimes\hat{v}-\hat{w}\otimes\hat{w})$ from the first term and adding it to the second, we find, setting $\hat{v}$ in the 1-direction and $\hat{w}$ in the second direction (note that for $\Theta>\pi/2$, the orientation of the $(\hat{w},\hat{v},\hat{r})$ basis changes), that

$$\mathbf{h}^{\rm TT}=\frac{2M_1M_2}{Mr}(\Omega a)^2\left[(1-|\cos\Theta|)^2\cos(2\Omega t)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+2|\cos\Theta|\begin{pmatrix}\cos(2\Omega t) & -\sin(2\Omega t)\\ -\sin(2\Omega t) & -\cos(2\Omega t).\end{pmatrix}\right] \tag{20}$$

The first term is a pure linear polarization, strongest when the binary is seen edge-on, and vanishing when it is seen face-on. The second term is a pure circular polarization, vanishing when the binary is seen edge-on, and strongest when it is seen face-on.