# General Relativity Fall 2018
# Lecture 18: Angular and linear momentum radiated by GWs. Introduction to the Post-Newtonian approximation

Yacine Ali-Haïmoud

### Angular momentum carried by GWs

Last time we gave the effective stress-energy tensor of GWs, $T^{\rm GW}_{\mu\nu} = \langle \partial_\mu h^{\rm TT}_{mn} \partial_\nu h^{\rm TT}_{mn}\rangle/(32\pi)$. From this we derived the rate of energy loss of a source with a time-varying mass quadrupole moment, the quadrupole formula, $dE/dt = \langle \dddot{Q}_{ij} \dddot{Q}_{ij}\rangle/5$. We can similiarly compute the rate of change of angular momentum:

$$J_i \equiv \epsilon_{ijk} \int d^3x \ x^j \tau^{0k} \quad \Rightarrow \frac{dJ_i}{dt} = \epsilon_{ijk} \int d^3x \ x^j \partial_0 \tau^{0k} = -\epsilon_{ijk} \int d^3x \ x^j \partial_l \tau^{lk} = -\epsilon_{ijk} \int_S dS_l \ x^j \tau^{lk}, \tag{1}$$

where the last integral is a surface integral, resulting from Stokes' theorem. Taking the surface far away from the sources, we may replace $\tau_{lk} \rightarrow T^{\rm GW}_k$.

Recall that $h^{\rm TT}_{mn} = (2/r)\mathcal{P}^{\rm TT}_{mnab} \ddot{Q}_{ab}(t-r)$. Hence, we find

$$\begin{aligned} \partial_k h^{\rm TT}_{mn} &= \frac{2}{r} \ddot{Q}_{ab}(t-r) \partial_k \mathcal{P}^{\rm TT}_{mnab} + \mathcal{P}^{\rm TT}_{mnab} \hat{x}^k \frac{\partial}{\partial r}\left(\frac{2}{r}\ddot{Q}_{ab}(t-r)\right) \\ &= \frac{2}{r} \ddot{Q}_{ab}(t-r) \partial_k \mathcal{P}^{\rm TT}_{mnab} - \hat{x}^k \mathcal{P}^{\rm TT}_{mnab}\left(\frac{2}{r}\dddot{Q}_{ab}(t-r) - \frac{2}{r^2}\ddot{Q}_{ab}(t-r)\right). \end{aligned} \tag{2}$$

The first term is of order $1/r^2$, but the second term vanishes when multiplied by $\epsilon_{ijk} x^j$. So we find, to leading order,

$$\epsilon_{ijk} x^j T^{\rm GW}_{lk} = -\frac{4}{r^2} \hat{x}^l \epsilon_{ijk} x^j \partial_k \mathcal{P}^{\rm TT}_{mnab} \mathcal{P}^{\rm TT}_{mncd} \langle \ddot{Q}_{ab} \dddot{Q}_{cd}\rangle. \tag{3}$$

Computing the integral over angles like we did for $\dot{M}$, we arrive at

$$\frac{dJ_i}{dt} = -\frac{2}{5}\epsilon_{ijk} \langle \ddot{Q}_{jl} \dddot{Q}_{kl}\rangle. \tag{4}$$

Let us apply this to a circular binary: there we had found (in that case $\ddot{\mathbf{Q}} = \ddot{\mathbf{I}}$)

$$\ddot{\mathbf{Q}} = \frac{2M_1M_2}{M}(\Omega a)^2 \left[\cos(2\Omega t)(\hat{v}\otimes\hat{v} - \hat{u}\otimes\hat{u}) - \sin(2\Omega t)(\hat{u}\otimes\hat{v} + \hat{v}\otimes\hat{u})\right], \tag{5}$$

$$\dddot{\mathbf{Q}} = -\frac{4M_1M_2}{M}\Omega^3 a^2 \left[\sin(2\Omega t)(\hat{v}\otimes\hat{v} - \hat{u}\otimes\hat{u}) + \cos(2\Omega t)(\hat{u}\otimes\hat{v} + \hat{v}\otimes\hat{u})\right], \tag{6}$$

so that

$$\ddot{Q}_{jl} \dddot{Q}_{kl} = -8\left(\frac{M_1M_2}{M}\right)^2 \Omega^5 a^4 \left(u_i v_k - v_j u\right), \tag{7}$$

so that

$$\frac{d\vec{J}}{dt} = -\frac{32}{5}\left(\frac{M_1M_2}{M}\right)^2 \Omega^5 a^4 \hat{J}. \tag{8}$$

This should give the same result for $\dot{\Omega}$ as we obtained for the rate of energy loss, implying that a circular orbit does not gain eccentricity due to GW radiation (which is sensible: otherwise what would generate the eccentricity's preferred direction?).

**Linear momentum loss due to GWs and higher-order multipoles**

We can similarly compute the rate of change of linear momentum,

$$P^i \equiv \int d^3x \ \tau^{0i} \quad \Rightarrow \frac{dP^i}{dt} = \int d^3x \ \partial_0 \tau^{0i} = -\int d^3x \ \partial_k \tau^{ki} = -\int_S dS_k \ \tau^{ki}. \tag{9}$$

To compute the surface integral, we just need the $1/r^2$ term in $T^{\rm GW}_{ki}$ (other terms contributed vanishingly small amounts as $r \rightarrow \infty$), that is

$$T^{\rm GW}_{ik} \approx \frac{1}{8\pi r^2} \mathcal{P}^{\rm TT}_{mnab} \mathcal{P}^{\rm TT}_{mncd} \hat{x}^i \hat{x}^k \langle \dddot{Q}_{ab} \dddot{Q}_{cd} \rangle. \tag{10}$$

When computing the angle integral, we find that it is proportional to the angle-average of $\mathcal{P}^{\rm TT}_{mncd} \hat{x}^i$. The result must be an isotropic tensor (i.e. built only out of Kronecker deltas), but it has an odd number (five) of indices! So the angle average vanishes.

Hence, the mass quadrupole contribution to $h^{\rm TT}_{ij}$ leads to no net loss of linear momentum. Still, linear momentum can be radiated by GWs, just at a higher-order in the characteristic velocities in the sources. Remember that the actual solution for $h^{\rm TT}_{ij}$ is

$$h^{\rm TT}_{ij}(t, \vec{x}) = 4 \mathcal{P}^{\rm TT}_{ijkl} \int d^3y \frac{T_{kl}(t - |\vec{x} - \vec{y}|, \vec{y})}{|\vec{x} - \vec{y}|} \approx \frac{4}{r} \mathcal{P}^{\rm TT}_{ijkl} \int d^3y \ T_{kl}(t - r + \hat{x} \cdot \vec{y}, \vec{y}) + \mathcal{O}(1/r^2). \tag{11}$$

The Quadrupole term came from neglecting $\hat{x} \cdot \vec{y}$ inside $T_{ij}$. But we could Taylor-expand to get more terms, each one proportional to a higher power of $y \partial_0 \sim V$. The strain is therefore of the form (see e.g. Thorne & Blandford, Chapter 27, and Thorne 1980)

$$h \sim \frac{1}{r} \left[ \sum_{\ell \geq 2} \alpha_\ell \ \partial_t^\ell (\mathcal{M}_\ell) + \beta_\ell \ \partial_t^\ell (\mathcal{C}_\ell) \right], \tag{12}$$

where $\mathcal{M}_\ell \sim M L^\ell$ is the mass $\ell$-th multipole and $\mathcal{C}_\ell \sim M v L^\ell$ is the mass current $\ell$-th multipole. In terms of power of characteristic velocity,

$$\partial_t^\ell (\mathcal{M}_\ell) \sim M v^\ell, \quad \partial_t^\ell (\mathcal{C}_\ell) \sim M v^{\ell+1}. \tag{13}$$

This is analogous to the multipole expansion of electromagnetic waves. Linear momentum can indeed be radiated by gravitational waves but, to lowest order, with a rate proportional to $\dddot{\mathcal{M}}_2 \times \dddot{\mathcal{C}}_2$ and $\dddot{\mathcal{M}}_2 \times \ddddot{\mathcal{M}}_3$. This linear momentum radiation leads to a recoil velocity for the final product of a merger, which can be several hundreds of km/s.

## INTRODUCTION TO THE POST-NEWTONIAN APPROXIMATION

Consider an ensemble of self-gravitating particles with characteristic total mass $M$, characteristic separations $L$, and characteristic velocities $v \sim \sqrt{M/L}$. To lowest order in velocities, we know since Newton that slowly-moving particles evolve according to

$$\frac{d\vec{v}}{dt} = -\vec{\nabla}\phi, \tag{14}$$

$$\nabla^2 \phi = 4\pi\rho. \tag{15}$$

The right-hand-side of Newton's equation if of order $\phi/L \sim (M/L)/L \sim v^2/L$. The goal of the post-Newtonain expansion is to derive equations of motion for particles to increasingly high powers of velocity. This can be seen as an expansion in powers of $c^{-2}$ or $G$. We follow closely the treatment of Weinberg's textbook: we first establish to which order in $v$ we need the Christoffel symbols, then determine the needed metric coefficients from the Einstein equation, which we also expand in powers of the characteristic velocity.

### Geodesic equation

Let us rewrite the geodesic equation in terms of coordinate time, and in terms of velocities:

$$\begin{aligned}\frac{d^2x^i}{dt^2} &= \frac{1}{dt/d\tau}\frac{d}{d\tau}\left[\frac{1}{dt/d\tau}\frac{dx^i}{d\tau}\right] = \frac{1}{(dt/d\tau)^2}\left(\frac{d^2x^i}{d\tau^2} - \frac{dx^i/d\tau}{dt/d\tau}\frac{d^2t}{d\tau^2}\right)\\ &= \frac{1}{(dt/d\tau)^2}\left(-\Gamma^i_{\mu\nu} + \frac{dx^i}{dt}\Gamma^0_{\mu\nu}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \left(-\Gamma^i_{\mu\nu} + \frac{dx^i}{dt}\Gamma^0_{\mu\nu}\right)\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\\ &= -\Gamma^i_{00} + \left[\Gamma^0_{00}v^i - 2\Gamma^i_{0j}v^j\right] + \left[2\Gamma^0_{0j}v^iv^j - \Gamma^i_{jk}v^jv^k\right] + \Gamma^0_{jk}v^iv^jv^k. \end{aligned} \tag{16}$$

The Newtonian approximation is to keep only the first term, $-\Gamma^i_{00} = -\partial_i\Phi \sim M/L^2 \sim v^2/L$. The 1st post-Newtonian correction consists in computing contributions up to order $v^4/L$. We see that we need Christoffel symbols at different orders: we need

$$\Gamma^i_{00} \quad \text{to order} \quad v^4/L, \tag{17}$$
$$\Gamma^0_{00} \text{ and } \Gamma^i_{0j} \quad \text{to order} \quad v^3/L, \tag{18}$$
$$\Gamma^0_{0j} \text{ and } \Gamma^i_{jk} \quad \text{to order} \quad v^2/L, \tag{19}$$
$$\Gamma^0_{jk} \quad \text{to order} \quad v/L, \tag{20}$$

### Metric coefficients

We write the metric coefficients as a series expansion in orders in velocity.

$$g_{\mu\nu} = \eta_{\mu\nu} + \sum_N {}^N g_{\mu\nu}, \qquad {}^N g_{\mu\nu} \sim v^N \sim (M/L)^{N/2}. \tag{21}$$

Upon flipping spatial coordinates, $\vec{x} \to -\vec{x}$, the velocities change sign, $g_{00}$ and $g_{ij}$ remain unchanged, and $g_{0i}$ changes sign. We will therefore only have even $N$ for $g_{00}$ and $g_{ij}$ and odd ones for $g_{0i}$. Moreover, we already computed the lowest-order contribution to $g_{0i} \sim J/r^2 \sim Mrv/r^2 \sim Mv/r \sim v^3$. So we expect the following expansion:

$$g_{00} = -1 + {}^2g_{00} + {}^4g_{00} + ..., \tag{22}$$
$$g_{0i} = {}^3g_{0i} + {}^5g_{0i} + ..., \tag{23}$$
$$g_{ij} = \delta_{ij} + {}^2g_{ij} + {}^4g_{ij} + ... \tag{24}$$

### Christoffel symbols

Let us now compute Chirstoffel symbols ${}^N\Gamma^\lambda_{\mu\nu} \sim v^N/L$. Remember that in slow motion, $\partial_t \sim v/ \sim v\partial_x$. First, there are no metric coefficients at order $v$ (they all start at least at order $v^2$), hence we conclude that ${}^1\Gamma^0_{jk} = 0$.

Next, to compute Chirstoffel symbols at order $N \leq 3$, we may set the inverse-metric prefactor to the inverse Minkowski metric (since the remainder is already at least at order 2, and including perturbations to the inverse metric would lead to terms of order 4). We then get

$${}^2\Gamma^0_{0j} = -\frac{1}{2}\,{}^2\left(g_{00,j} + g_{j0,0} - g_{0j,0}\right) = -\frac{1}{2}\,{}^2g_{00,j}, \tag{25}$$
$${}^2\Gamma^i_{jk} = \frac{1}{2}\left({}^2g_{ij,k} + {}^2g_{ki,j} - {}^2g_{jk,i}\right), \tag{26}$$
$${}^2\Gamma^i_{00} = -\frac{1}{2}\,{}^2g_{00,i} \tag{27}$$
$${}^3\Gamma^0_{00} = \frac{1}{2}\,{}^2g_{00,0}, \tag{28}$$
$${}^3\Gamma^i_{0j} = \frac{1}{2}\left({}^3g_{0i,j} - {}^3g_{0j,i} + {}^2g_{ij,0}\right). \tag{29}$$

Finally, at order $N = 4$, we do need the second-order correction to the metric inverse. We find

$${}^4\Gamma^i_{00} = -\frac{1}{2}\left({}^4g_{00,i} + {}^2g^{ij}\,{}^2g_{00,j}\right) + {}^3g_{0i,0}. \tag{30}$$

### Ricci tensor

We now build the Ricci tensor as a series $R_{\mu\nu} = \sum_N {}^N R_{\mu\nu}$, with ${}^N R_{\mu\nu} \sim v^N/L^2$. To simplify we use the harmonic gauge condition

$$g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = -\frac{1}{\sqrt{|g|}}\partial_\sigma\left(\sqrt{|g|}g^{\lambda\sigma}\right) = 0, \tag{31}$$

which generalizes the Lorenz gauge in linearized gravity, $\partial_\sigma \overline{h}^{\lambda\sigma} = 0$.

One then computes the components of the Ricci tensor in that gauge. For instance,

$$^2R_{00} = \frac{1}{2}\nabla^2(^2g_{00}), \tag{32}$$

$$^4R_{00} = \frac{1}{2}\nabla^2(^2g_{00}) - \frac{1}{2}\partial_t^2(^2g_{00}) - \frac{1}{2}\ {}^2g^{ij}\partial_i\partial_j(^2g_{00}) + \frac{1}{2}\left(\nabla^2\ {}^2g_{00}\right)^2 \tag{33}$$

### Stress-energy tensor

Finally, we also expand the stress-energy tensor in power of velocity, $T_{\mu\nu} = \sum_N {}^N T_{\mu\nu}$, with ${}^N T_{\mu\nu} \sim v^N/L^2$. Note that here I break from Weinberg's convention, so that we can have the same order $N$ on each side of the Einstein field equation. This implies that $T_{\mu\nu}$ starts at order $N = 2$:

$$T_{00} = {}^2T_{00} +^4 T_{00} + ..., \tag{34}$$

$$T_{0i} = {}^3T_{0i} + ..., \tag{35}$$

$$T_{ij} = {}^4T_{ij} + ..., \tag{36}$$

where the last equation comes from the fact that $T_{ij} \sim \rho v^2$ if of order $v^4$. The lowest order term is the rest-mass density, ${}^2T^{00}$. The term ${}^4T^{00}$ contains the kinetic energy correction to the energy density.

Solving Einstein's field equations order-by-order in $v^N$ gives equations satisfied by the metric coefficients, order by order, and generalizing Poisson's equation.

### Einstein-Infeld-Hoffmann equation

Combining the geodesic equation with the Einstein equation, we arrive at the generalization of the Newton-Poisson system, valid at order $v^4$, and known as the Einstein-Infeld-Hoffmann equations:

$$\frac{d\vec{v}}{dt} = -(1 + v^2 + 4\phi)\vec{\nabla}\phi - \vec{\nabla}\psi - \partial_t\vec{\xi} + \vec{v}\times(\vec{\nabla}\times\vec{\xi}) + \left(3\partial_t\phi + 4\vec{v}\cdot\vec{\nabla}\phi\right)\vec{v}, \tag{37}$$

$$\Delta\phi = 4\pi\ {}^2T^{00}, \tag{38}$$

$$\Delta\psi = \partial_t^2\phi + 4\pi\left({}^4T^{00} +^4 T^{ii}\right), \tag{39}$$

$$\Delta\xi^i = 16\pi\ {}^3T^{0i}. \tag{40}$$

In terms of metric coefficients, we have

$$ds^2 = -(1 + 2\phi + 2\psi + 2\phi^2)dt^2 + 2\xi_i\ dtdx^i + (1 - 2\phi)\delta_{ij}dx^i dx^j, \tag{41}$$

where we did not need to give the ${}^4g_{ij}$ term as it does not contribute to that order (as can be seen from expressions of the Christoffel symbols).

We have already encountered the $\vec{v}\times(\vec{\nabla}\times\vec{\xi})$ term in our study of stationary sources: this is the frame-dragging term, or gravito-magnetic term, leading to Lense-Thirring precession.