General Relativity Fall 2018
Lecture 19: Symmetries, spherically-symmetric spacetimes

Yacine Ali-Haïmoud

There are two regimes where GR has known analytic solutions: either in the weak-gravity regime, which we have studied so far, or in the case of highly symmetric spacetimes, on which we will now focus.

The notion of symmetry is conveyed by **Killing vector fields** (see Homework 5). These are vector fields ζ^μ that satisfy Killing's equation $\zeta_{(\mu;\nu)} = \nabla_{(\nu}\zeta_{\mu)} = 0$. If the Killing vector field $\zeta = \partial_{(\sigma*)}$ is the partial derivative operator with respect to some coordinate $\sigma*$, then, in a coordinate system that has $x^{\sigma*}$ as one of the coordinates, the metric components do not depend on $x^{\sigma*}$, i.e. $\partial_{\sigma*} g_{\mu\nu} = 0$. We also saw that the scalar curvature does not change along curves tangent to a Killing vector field (HW5).

Symmetries of flat spacetime

Let us now derive what are the Killing fields of flat spacetime. Let us use a globally inertial coordinate system, i.e. coordinates in which the metric tensor is the Minkowski metric everywhere: $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$. In these flat coordinates, Killing's equation becomes $\partial_{(\mu}\zeta_{\nu)} = 0$, i.e. $\zeta_{\nu,\mu} = -\zeta_{\mu,\nu}$. Let us define the antisymmetric tensor field $f_{\mu\nu} \equiv \zeta_{\mu,\nu}$. Taking a derivative, we get

$$F_{\mu\nu\sigma} \equiv f_{\mu\nu,\sigma} = \zeta_{\mu,\nu\sigma} = \zeta_{\mu,\sigma\nu} = F_{\mu\sigma\nu}. \tag{1}$$

Therefore the tensor $F_{\mu\nu\sigma}$ is antisymmetric in its first two indices and symmetric in its last two. Flipping the first two and last two consecutively, implies that moving indices to the left by one cyclic permutation changes the sign:

$$F_{abc} = -F_{bac} = -F_{bca}. \tag{2}$$

Do this cyclic permutation three times to get back to the same indices:

$$F_{abc} = -F_{bca} = +F_{cab} = -F_{abc}. \tag{3}$$

This implies $F_{abc} = 0 = f_{ab,c}$. Hence, for each a, b, f_{ab} is constant. Integrating one more time, we find

$$\zeta_\mu = f_{\mu\nu} x^\nu + b_\mu, \tag{4}$$

where b_μ is a constant. Killing vector fields of flat spacetime are therefore entirely determined by 10 constants: the 4 constants b_μ, and the 6 components of the antisymmetric tensor $f_{\mu\nu}$.

Let us now see what each one of these components represents.

Translations

First, consider the Killing vector fields with $f_{\mu\nu} = 0$. They are linear combinations of the coordinate basis vectors, which are hence themselves Killing fields:

$$T_{(\mu)} \equiv e_{(\mu)} = \partial_{(\mu)}. \tag{5}$$

The fact that geometry does not change along Minkowski coordinate basis vectors means that flat spacetime is **symmetric under translations**. The vector fields $T_{(\mu)}$ are the generators of translations.

Rotations

Now consider Killing fields which have $b_\mu = 0$, and $f_{0\nu} = 0$, i.e. linear combinations of the three fields $J_{(j)}$ with components

$$J^0_{(j)} = 0, \quad J^i_{(j)} \equiv \epsilon_{ijk} x^k, \tag{6}$$

i.e., explicitly,

$$J_{(3)} = x^1 \partial_{(2)} - x^2 \partial_{(1)}, \tag{7}$$

and $J_{(1)}$ and $J_{(2)}$ are similarly defined by cyclic permutations. These fields are associated with **rotational symmetry**. To see this, define spherical polar coordinates, with $e_{(3)}$ along the polar axis:

$$x^1 = r \sin\theta \cos\varphi, \qquad x^2 = r \sin\theta \sin\varphi, \qquad x^3 = \cos\theta. \tag{8}$$

We then have

$$\frac{\partial}{\partial\varphi} = \frac{\partial x^i}{\partial\varphi} \frac{\partial}{\partial x^i} = -x^2 \partial_{(1)} + x^1 \partial_{(2)} = J_{(3)}. \tag{9}$$

Therefore, the fact that $J_{(3)}$ is a Killing vector field is due to the symmetry of spacetime under change of φ, i.e. rotations about $e_{(3)}$, and similarly for $J_{(1)}$ and $J_{(2)}$. Said differently, $J_{(i)}$ is the **generator of rotations** about $e_{(i)}$. Note that, in the smae spherical polar coordinates, we have

$$J_{(1)} = -\sin\varphi \partial_\theta - \text{cotan}\theta \cos\varphi \partial_\varphi, \qquad J_{(2)} = \cos\varphi \partial_\theta - \text{cotan}\theta \sin\varphi \partial_\varphi. \tag{10}$$

Therefore, even though the 3 $J_{(1)}$ are linearly-independent vector *fields*, they only span a 2-dimensional space at each point of the manifold, since they are combinations of ∂_θ and ∂_φ only.

Boosts

Finally, consider Killing fields with $b_\mu = 0$ and $f_{ij} = 0$, i.e. linear combinations of the three fields $K_{(i)}$ defined as

$$K_{(i)} = x^i \partial_{(0)} + x^0 \partial_{(i)}. \tag{11}$$

One can check that this corresponds to $f_{i0} = -f_{0i} = 1$. These fields are the **generators of Lorentz boosts**. To see this, write $x^0 = X^0 \cosh\psi + X^1 \sinh\psi$, $x^1 = X^1 \cosh\psi + X^0 \sinh\psi$, so that the coordinates X^μ are obtained from the coordinates x^μ by a Lorentz boost of velocity $v = \tanh\psi$. We then have

$$\frac{\partial}{\partial\psi} = \frac{\partial x^i}{\partial\psi} \frac{\partial}{\partial x^i} = x^1 \partial_{(0)} + x^0 \partial_{(1)} = K_{(1)}. \tag{12}$$

Therefore, the fact that $K_{(1)}$ is a Killing vector field is due to the symmetry of spacetime under boosts along $e_{(1)}$, and similarly for $K_{(2)}$ and $K_{(3)}$.

To conclude, flat spacetime has 10 independent Killing vector fields: the 4 generators of translations, the 3 generators of spatial rotations, and the 3 generators of Lorentz boosts. Of course, any linear combination of these remains a Killing vector field.

Commutation relations of rotations and translations in flat spacetime

Recall that the commutator of two vector fields is the vector field with components

$$[X, Y]^\mu = X^\nu \nabla_\nu Y^\mu - Y^\nu \nabla_\nu X^\mu = X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu, \tag{13}$$

where the second relation is always true, since the Christoffel symbols cancel out.

We therefore immediately immediately find that $[T_{(\mu)}, T_{(\nu)}] = 0$. Since spatial rotations do not involve the time component, we moreover find that $[J_{(i)}, T_{(0)}] = 0$. Let us now compute the other commutators.

$$[J_{(m)}, J_{(n)}]^0 = 0 \tag{14}$$

$$\begin{aligned}
[J_{(m)}, J_{(n)}]^i &= J^j_{(m)} \partial_j J^i_{(n)} - J^j_{(n)} \partial_j J^i_{(m)} = \epsilon_{jmk} x^k \epsilon_{inl} \partial_j x^l - \epsilon_{jnk} x^k \epsilon_{iml} \partial_j x^l = (\epsilon_{jmk}\epsilon_{inl} - \epsilon_{jnk}\epsilon_{iml})\, x^k \delta^l_j \\
&= (\epsilon_{jmk}\epsilon_{inj} - \epsilon_{jnk}\epsilon_{imj})\, x^k = (\delta_{mi}\delta_{kn} - \delta_{mn}\delta_{ki} - \delta_{ni}\delta_{km} + \delta_{mn}\delta_{ki})\, x^k = (\delta_{mi}\delta_{nk} - \delta_{ni}\delta_{mk})\, x^k \\
&= \epsilon_{jmn}\epsilon_{jik} x^k = -\epsilon_{jmn} J^i_{(j)}.
\end{aligned} \tag{15}$$

Now consider commutations with spatial translations:

$$[J_{(m)}, T_{(n)}]^0 = 0, \tag{16}$$

$$[J_{(m)}, T_{(n)}]^i = J^j_{(m)}\partial_j T^i_{(n)} - T^j_{(n)}\partial_j J^i_{(m)} = -\delta^j_n \epsilon_{imk}\delta^k_j = -\epsilon_{imn} = -\epsilon_{mnk}T^i_{(k)}. \tag{17}$$

To summarize, we therefore have found the following commutation relations:

$$[J_{(m)}, J_{(n)}] = -\epsilon_{mnj}J_{(j)}, \qquad [J_{(m)}, T_{(n)}] = -\epsilon_{mnj}T_{(j)}, \tag{18}$$

Formal definition stationarity, spherical symmetry, homogeneity

Based on the properties of Killing vector fields in flat spacetime, we can now define formally symmetry properties in an aribitrary spacetime.

• A spacetime is **stationary** if it possesses a Killing vector field $T_{(0)}$ that is timelike in some portion of spacetime. In practice, this notion is only really useful for asymptotically flat spacetimes, in which case, the aforementioned Killing vector field should be timelike in the asymptotically flat region of spacetime.

• A spacetime is **spherically symmetric** if it possesses 3 spacelike Killing vector fields $J_{(1)}, J_{(2)}, J_{(3)}$ which satisfy the commutation relations

$$[J_{(i)}, J_{(j)}] = -\epsilon_{ijk}J_{(k)}. \tag{19}$$

• A spacetime is **homogeneous and isotropic** if it is spherically symmetric, with 3 rotation generators $J_{(i)}$, *and* possesses 3 additional spacelike Killing vector fields $T_{(1)}, T_{(2)}, T_{(3)}$ such that

$$[T_{(i)}, T_{(j)}] = 0, \quad [J_{(i)}, T_{(j)}] = -\epsilon_{ijk}T_{(k)}. \tag{20}$$

Metric of a spherically symmetric spacetime

Using the general definition of spherical symmetry, we now derive the general form of the metric of a spherically-symmetric spacetime. We will restrict ourselves to spherically-symmetric spacetimes for which, like in flat spacetime, the generators of rotation $J_{(i)}$ only span a 2-dimensional vector space at each point of the manifold. Note that they are still linearly independent vector *fields*. This restriction means that we do not consider spaces like the three-sphere, for instance. We expect the metric of such spacetimes can be cast in the form

$$\boxed{ds^2 = g_{tt}(t,r)dt^2 + g_{rr}(t,r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)}. \tag{21}$$

We now derive this formally, following C. M. Hirata's lecture notes.

As a preliminary, note that for any two vector fields X, Y and scalar function f,

$$[X, fY] = f[X,Y] + (X^\nu\partial_\nu f)Y. \tag{22}$$

• By assumption, the $J_{(i)}$ are linearly dependent at each point of spacetime. As a consequence, they span at most a 2-dimensional vector space at each point. Let us now show that they cannot span a one-dimensional manifold over a finite region of the manifold. Indeed, if this were the case, it would mean that, locally, the $J_{(i)}$'s are all proportional to a single vector field, say $J_{(3)} \equiv d/d\lambda$: $J_{(1)} = A_1(P)J_{(3)}$, $J_{(2)} = A_2(P)J_{(3)}$. We would then have

$$[J_{(3)}, J_{(1)}] = [J_{(3)}, A_1 J_{(3)}] = (J^\mu_{(3)}\partial_\mu A_1)J_{(3)} = \frac{dA_1}{d\lambda}J_{(3)}. \tag{23}$$

On the other hand, this is equal to $-J_{(2)} = -A_2 J_{(3)}$. So we find $dA_1/d\lambda = -A_2$. Similarly, computing $[J_{(3)}, J_{(2)}]$ two different ways gives us $dA_2/d\lambda = +A_1$. Finally, computing $[J_{(1)}, J_{(2)}]$ two different ways implies $A_1(dA_2/d\lambda) - A_2(dA_1/d\lambda) = -1$. Substituting, we get $(A_1)^2 + (A_2)^2 = -1$, which has no solution. Hence we conclude that **the $J_{(i)}$ cannot span a 1-dimensional vector space except at punctual events of the manifold**.

•. Given a point P, we define the submanifold $S(P) \in \mathcal{M}$ made of all curves passing by P and whose tangent vector is a linear combination of the $J_{(i)}$. Except at peculiar P's, this is a 2-dimensional manifold (to build intuition, these $S(P)$ are just the 2-spheres of constant t and r).

We label different submanifolds $S(P)$ by two coordinates x^0, x^1, constant on each $S(P)$.

Since the $J_{(i)}$ are linearly dependent, it means that one can find three scalar functions $c_1(P), c_2(P), c_3(P)$ such that, at each point P of spacetime, $\sum_{i=1}^3 c_i J_{(i)} = 0$. We may rescale these functions by $\sum_i c_i^2$, and, at each P, define two numbers $\theta(P), \varphi(P)$ such that $c_1(P) = \sin\theta\cos\varphi,\ c_2(P) = \sin\theta\sin\varphi,\ c_3(P) = \cos\theta$, i.e.

$$\sin\theta(\cos\varphi J_{(1)} + \sin\varphi J_{(2)}) + \cos\theta J_{(3)} = 0. \tag{24}$$

Assuming the $J'_{(i)}s$ are continuous vector fields, we can also define θ and φ continuously over each submanifold $S(P)$. We can moreover define them continuously over the entire manifold, enforcing that they are continuously defined from a two-sphere $S(P)$ to the next. We thus have defined 4 coordinates, $x^0, x^1, \theta, \varphi$.

The tangent vector of any curve in $S(P)$ has components $dx^\mu/d\lambda$. Given that the coordinates x^0, x^1 are constant on each $S(P)$, any vector tangent to $S(P)$ must have vanishing 0 and 1 contravariant components. This is therefore the case for the $J_{(i)}$, which take the form

$$J_{(i)} = J^\theta_{(i)}\partial_\theta + J^\varphi_{(i)}\partial_\varphi. \tag{25}$$

• Let us define the vector field $V \equiv \sin\theta(\cos\varphi J_{(1)} + \sin\varphi J_{(2)}) + \cos\theta J_{(3)}$. This vector field is zero everywhere, by definition of the angles θ, φ. Compute its commutator with $J_{(3)}$, using Eq. (22):

$$\begin{aligned} 0 &= [J_{(3)}, V] \\ &= \sin\theta\cos\varphi[J_{(3)}, J_{(1)}] + \left[J^\mu_{(3)}\partial_\mu\left(\sin\theta\cos\varphi\right)\right] J_{(1)} + \sin\theta\sin\varphi[J_{(3)}, J_{(2)}] + \left[J^\mu_{(3)}\partial_\mu\left(\sin\theta\sin\varphi\right)\right] J_{(2)} + \left[J^\mu_{(3)}\partial_\mu\cos\theta\right] J_{(3)}. \end{aligned}$$

Express the commutators with the assumed commutation relations, and multiply everything by $\cos\theta$ to eliminate $J_{(3)}$ by using Eq. (24):

$$\begin{aligned} 0 = &\left(\cos\theta\sin\theta\sin\varphi + \cos\theta J^\mu_{(3)}\partial_\mu\left(\sin\theta\cos\varphi\right) - \sin\theta\cos\varphi\ J^\mu_{(3)}\partial_\mu\cos\theta\right) J_{(1)} \\ &+ \left(-\cos\theta\sin\theta\cos\varphi + \cos\theta J^\mu_{(3)}\partial_\mu\left(\sin\theta\sin\varphi\right) - \sin\theta\sin\varphi\ J^\mu_{(3)}\partial_\mu\cos\theta\right) J_{(2)}. \end{aligned} \tag{26}$$

Substituting Eq. (25), we can simplify this expression substantially, and get

$$0 = \left(\cos\theta\sin\theta\sin\varphi\left(1 - J^\varphi_{(3)}\right) + \cos\varphi J^\theta_{(3)}\right) J_{(1)} + \left(\cos\theta\sin\theta\cos\varphi\left(J^\varphi_{(3)} - 1\right) + \sin\varphi J^\theta_{(3)}\right) J_{(2)}. \tag{27}$$

Since $J_{(1)}$ and $J_{(2)}$ are linearly independent at a generic point (otherwise the three vectors would only span a 1-dimensional vector space, which we showed is not allowed), we conclude that

$$J^\theta_{(3)} = 0, \quad J^\varphi_{(3)} = 1 \qquad \Rightarrow J_{(3)} = \partial_\varphi. \tag{28}$$

Simliarly, computing $[J_{(1)}, V]$ and $[J_{(2)}, V]$ lead to

$$J_{(1)} = -\sin\varphi\partial_\theta - \text{cotan}\theta\cos\varphi\partial_\varphi, \qquad J_{(2)} = \cos\varphi\partial_\theta - \text{cotan}\theta\sin\varphi\partial_\varphi. \tag{29}$$

• We now, and finally, use the fact that the $J_{(i)}$'s are Killing vector fields. Let us explicitly write Killing's equation:

$$\begin{aligned} 0 &= \zeta_{\mu;\nu} + \zeta_{\nu;\mu} = g_{\lambda\mu}\zeta^\lambda{}_{;\nu} + g_{\lambda\nu}\zeta^\lambda{}_{;\mu} = g_{\lambda\mu}\zeta^\lambda{}_{,\nu} + g_{\lambda\nu}\zeta^\lambda{}_{,\mu} + \left(g_{\lambda\mu}\Gamma^\lambda_{\nu\sigma} + g_{\lambda\nu}\Gamma^\lambda_{\mu\sigma}\right)\zeta^\sigma \\ &= g_{\lambda\mu}\zeta^\lambda{}_{,\nu} + g_{\lambda\nu}\zeta^\lambda{}_{,\mu} + g_{\mu\nu,\sigma}\zeta^\sigma, \end{aligned} \tag{30}$$

where the second line is obtained from explicitly re-writing the Christoffel symbols.

Applying this equation to $\zeta = J_{(3)} = \partial_\varphi$, we find that all metric components are independent of φ:

$$\boxed{g_{\mu\nu,\varphi} = 0}, \quad \forall\ \mu, \nu. \tag{31}$$

Let us now apply the Killing equation to $J_{(1)}$ and $J_{(2)}$:

$$0 = g_{\lambda\mu}J^\lambda_{(i),\nu} + g_{\lambda\nu}J^\lambda_{(i),\mu} + g_{\mu\nu,\theta}J^\theta_{(i)}, \tag{32}$$

where we used $g_{\mu\nu,\varphi} = 0$.

Let us first apply this equation for $\mu, \nu \in \{0, 1\}$. Since $J^\lambda_{(i),0} = J^\lambda_{(i),1} = 0$, we obtain

$$\boxed{g_{00,\theta} = g_{01,\theta} = g_{11,\theta} = 0}. \tag{33}$$

i.e. the $0-1$ part of the metric components do not depend on θ.

Let us now apply this equation with $\nu = a \in \{0,1\}$ and $\mu = \theta$:

$$0 = g_{a\lambda} J^{\lambda}_{(i),\theta} + g_{a\theta,\theta} J^{\theta}_{(i)} = g_{a\varphi} J^{\varphi}_{(i),\theta} + g_{a\theta,\theta} J^{\theta}_{(i)} \tag{34}$$

since $J^{\theta}_{(i),\theta} = 0$ for all $J_{(i)}$. Apply this relation to $J_{(1)}$, at $\varphi = 0$, and find $g_{a\varphi}(\varphi = 0) = 0$. Since $g_{a\varphi}$ is independent of φ, we conclude that, for all φ,

$$\boxed{g_{0\varphi} = g_{1\varphi} = 0}. \tag{35}$$

Now apply Killing's equation to $\nu = a \in \{0,1\}$ and $\mu = \varphi$:

$$0 = g_{\lambda a} J^{\lambda}_{(i),\varphi} + g_{a\varphi,\theta} J^{\theta}_{(i)} = g_{a\theta} J^{\theta}_{(i),\varphi}, \tag{36}$$

where we used $g_{a\varphi} = 0$ We then get

$$\boxed{g_{0\theta} = g_{1\theta} = 0}. \tag{37}$$

Now apply Killing's equation to $\mu = \nu = \theta$:

$$0 = 2g_{\lambda\theta} J^{\lambda}_{(i),\theta} + g_{\theta\theta,\theta} J^{\theta}_{(i)} = 2g_{\varphi\theta} J^{\varphi}_{(i),\theta} + g_{\theta\theta,\theta} J^{\theta}_{(i)}. \tag{38}$$

Evaluate this at $\varphi = 0$ and then $\varphi = \pi/2$ and obtain

$$\boxed{g_{\varphi\theta} = 0 - g_{\theta\theta,\theta}}. \tag{39}$$

Finally, evaluate Killing's equation at $\mu = \theta$, $\nu = \varphi$:

$$0 = g_{\lambda\theta} J^{\lambda}_{(i),\varphi} + g_{\lambda\varphi} J^{\lambda}_{(i),\theta} + g_{\theta\varphi,\sigma} J^{\sigma}_{(i)} = g_{\theta\theta} J^{\theta}_{(i),\varphi} + g_{\varphi\varphi} J^{\varphi}_{(i),\theta}. \tag{40}$$

Apply this to either $J_{(1)}$ or $J_{(2)}$ and get

$$\boxed{g_{\varphi\varphi} = \sin^2\theta \; g_{\theta\theta}}. \tag{41}$$

To summarize, we have found coordinates in which the metric takes the form

$$ds^2 = g_{00}(x^0,x^1)(dx^0)^2 + 2g_{01}(x^0,x^1)dx^0dx^1 + g_{11}(x^0,x^1)(dx^1)^2 + g_{\theta\theta}(x^0,x^1)\left[d\theta^2 + \sin^2\theta d\varphi^2\right]. \tag{42}$$

Since, by assumption, $J_{(3)} = \partial_\varphi$ is spacelike, we deduce that $g_{\theta\theta} > 0$. Let us define the new coordinate $r \equiv \sqrt{g_{\theta\theta}(x^0,x^1)}$, and define $\tilde{t}$ as another non-angular coordinate:

$$ds^2 = g_{\tilde{t}\tilde{t}}(\tilde{t},r)d\tilde{t}^2 + 2g_{\tilde{t}r}(\tilde{t},r)d\tilde{t}dr + g_{rr}(\tilde{t},r)dr^2 + r^2\left[d\theta^2 + \sin^2\theta d\varphi^2\right]. \tag{43}$$

Finally, we want to eliminate the cross term $g_{\tilde{t}r}$. Change to a new time coordinate $t(\tilde{t},r)$. The tr component of the metric becomes

$$g_{tr} = \frac{\partial\tilde{t}}{\partial t}\frac{\partial\tilde{t}}{\partial r}g_{\tilde{t}\tilde{t}} + \left(\frac{\partial\tilde{t}}{\partial t}\frac{\partial r}{\partial r} + \frac{\partial r}{\partial t}\frac{\partial\tilde{t}}{\partial r}\right)g_{\tilde{t}r} + \frac{\partial r}{\partial t}\frac{\partial r}{\partial r}g_{rr} = \frac{\partial\tilde{t}}{\partial t}\left(\frac{\partial\tilde{t}}{\partial r}g_{\tilde{t}\tilde{t}} + g_{\tilde{t}r}\right), \tag{44}$$

and the tt component becomes

$$g_{tt} = \left(\frac{\partial\tilde{t}}{\partial t}\right)^2 g_{\tilde{t}\tilde{t}} + \frac{\partial\tilde{t}}{\partial t}\frac{\partial r}{\partial t}g_{\tilde{t}r} + \left(\frac{\partial r}{\partial t}\right)^2 g_{rr} = \left(\frac{\partial\tilde{t}}{\partial t}\right)^2 g_{\tilde{t}\tilde{t}}. \tag{45}$$

In order for g_{tt} to not vanish, but g_{tr} to vanish, and assuming $g_{\tilde{t}\tilde{t}} \neq 0$, we set

$$\frac{\partial\tilde{t}}{\partial r} = -\frac{g_{\tilde{t}r}}{g_{\tilde{t}\tilde{t}}}. \tag{46}$$

This concludes our derivation of the metric (21). Note that the coordinate r should *not* be interpreted as the "distance" to some "origin": its only clear meaning is that spacelike surfaces of constant t and r (the spheres $S(P)$) have an area $4\pi r^2$, assuming θ and φ span the usual range of spherical polar coordinates.