General Relativity Fall 2018
Lecture 20: The Schwarzschild metric

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We have shown that the metric of a spherically-symmetric spacetime (defined by having 3-Killing vector fields which satisfy commutation relations $[J_{(i)}, J_{(j)}] = -\epsilon_{ijk} J_{(k)}$) can be cast in the form

$$ds^2 = g_{tt}(t, r) dt^2 + g_{rr}(t, r) dr^2 + r^2 d\Omega^2, \qquad d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2. \tag{1}$$

It is straightforward, if a bit painful, to compute the Riemann tensor of this metric. Without doing any computations, we can already guess some of its symmetry properties. First of all, spacetime is invariant under the transformation $\varphi \to -\varphi$. Under such a parity transformation, any component of the Riemann tensor with a single φ index flips sign, hence must vanish. The same is true for $\theta \to \pi - \theta$, which implies that components of the Riemann tensor with a single θ also vanish. In an orthonormal basis (denoted by hats), we expect that $R_{a\hat{\theta}b\hat{\theta}} = R_{a\hat{\varphi}b\hat{\varphi}}$. The orthonormal basis vectors are

$$\hat{e}_\theta = \frac{1}{r}\partial_\theta, \quad \hat{e}_\varphi = \frac{1}{r\sin\theta}\partial_\varphi. \tag{2}$$

We then have

$$\begin{aligned} R_{a\varphi b\varphi} &= \textbf{Riemann}(\partial_a, \partial_\varphi, \partial_a, \partial_\varphi) = r^2\sin^2\theta\ \textbf{Riemann}(\partial_a, \hat{e}_\varphi, \partial_a, \hat{e}_\varphi) = r^2\sin^2\theta R_{a\hat{\varphi}b\hat{\varphi}} = r^2\sin^2\theta R_{a\hat{\theta}b\hat{\theta}} \\ &= \sin^2\theta R_{a\theta b\theta}. \end{aligned} \tag{3}$$

So there are only 5 components of the Riemann tensor that we need to compute, and all others follow from symmetries, and these are

$$R_{trtr}, \quad R_{t\theta t\theta}, \quad R_{r\theta r\theta}, \quad R_{t\theta r\theta}, \quad R_{\theta\varphi\theta\varphi}. \tag{4}$$

From these, we then obtain the Ricci tensor. For instance, an explicit calculation gives the following components:

$$R_{tr} = \frac{1}{r}\frac{\partial_t g_{rr}}{g_{rr}}, \tag{5}$$

$$R_{\theta\theta} = 1 - \frac{1}{g_{rr}}\left[1 - \frac{r}{2}\left(\frac{\partial_r g_{rr}}{g_{rr}} - \frac{\partial_r g_{tt}}{g_{tt}}\right)\right]. \tag{6}$$

Birkhoff's theorem and the Schwarzschild metric

Let us now consider the metric of a spherically-symmetric spacetime **in vacuum**. This does not mean that the entire spacetime must be empty: we simply focus on the vacuum regions, where $G_{\mu\nu}$ hence $R_{\mu\nu}$ vanishes. Setting $R_{tr} = 0$, we find from Eq. (5) that $\partial_t g_{rr} = 0$, i.e. $g_{rr}(r)$ is a function of r only. Setting $R_{\theta\theta} = 0$, and using Eq. (6), we see that $\partial_r g_{tt}/g_{tt} = \partial_r \ln|g_{tt}|$ must be a function of r only. Integrating, we find that $\ln|g_{tt}|$ is a sum of a function of t only and a function of r only: $\ln|g_{tt}| = A(t) + B(r)$, i.e.

$$g_{tt} = \pm \mathrm{e}^{A(t)}\mathrm{e}^{B(r)}. \tag{7}$$

We rescale t to get rid of $A(t)$, i.e. define $dt' \equiv \mathrm{e}^{\frac{1}{2}A(t)}dt$. This brings the metric to the following form:

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + r^2 d\Omega^2. \tag{8}$$

We see that in this form, ∂_t is a Killing vector field. Provided $g_{tt} < 0$ in the asymptotically flat region (which we will show shortly), we have therefore already shown a non-trivial result, that a spherically-symmetric spacetime in vacuum must also be stationary!

We have not yet used the tt and rr Einstein field equations. They can be combined to give

$$R_{rr} + \frac{g_{rr}}{g_{tt}}R_{tt} = \frac{1}{r}\frac{\partial_r(g_{tt}g_{rr})}{g_{tt}g_{rr}} = 0. \tag{9}$$

This implies that $g_{rr}g_{tt}$ is a constant. This constant must be negative if the metric is to have signature $(-1,1,1,1)$. We can simply rescale t by some overall multiplicative constant to make this constant -1, i.e. to have $g_{rr} = -1/g_{tt}$.

The last step is to finally solve for g_{tt}. We rewrite the $\theta\theta$ Einstein field equation by susbtituting $g_{rr} = -1/g_{tt}$:

$$0 = 1 + g_{tt}\left(1 + r\frac{\partial_r g_{tt}}{g_{tt}}\right) = 1 + g_{tt} + r\partial_r g_{tt} = 1 + \partial_r(r g_{tt}). \tag{10}$$

Integrate this equation to get

$$r g_{tt} = -r + 2M \qquad \Rightarrow g_{tt} = -\left(1 - \frac{2M}{r}\right). \tag{11}$$

where M is a constant of integration. We therefore arrive at the famous Schwarzschild metric

$$\boxed{ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2}. \tag{12}$$

To summarize, we have found that the metric of a vacuum spherically symmetric spacetime must take the Schwarzschild form (up to coordinate redefinitions of ocurse). This need not be the case inside of sources, of course. In particular, provided there is matter up to some $r = R > 2M$, we need not worry about the apparent divergence of g_{rr} at $r \to 2M$. We will get back to this apparent singularity later on.

For $r > 2M$, the Killing vector field ∂_t is timelike, hence this spacetime is stationary (that is, if spacetime was vacuum everywhere).

For $r \gg 2M$, the metric takes the following form:

$$ds^2 \approx -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 + \frac{2M}{r}\right)dr^2 + r^2 d\Omega^2. \tag{13}$$

This is *almost* the far-field metric of an object with mass M, except that the $(1+2M/r)$ does not multiply the angular part. To fix it, define $\overline{r} = r - M$, so that

$$1 \pm \frac{2M}{r} = 1 \pm \frac{2M}{\overline{r}} + \mathcal{O}(M/r^2), \tag{14}$$

$$r^2 = \overline{r}^2 + 2M\overline{r} + M^2 = \left(1 + \frac{2M}{\overline{r}} + \mathcal{O}(M/r^2)\right)\overline{r}^2. \tag{15}$$

With this new coordinate, the metric at $r \gg M$ can therefore be rewritten as

$$ds^2 \approx -\left(1 - \frac{2M}{\overline{r}}\right)dt^2 + \left(1 + \frac{2M}{\overline{r}}\right)\left(d\overline{r}^2 + \overline{r}^2 d\Omega^2\right), \tag{16}$$

which is indeed the far-field metric of a source with mass M.

Timelike geodesics of Schwarzschild

Consider a massive particle initially in the equatorial plane $\theta = \pi/2$, with $u^\theta = 0$ initially. By symmetry, it will remain in that plane. To show this explicitly, compute the θ component of the geodesic equation:

$$\frac{du^\theta}{d\tau} = -\Gamma^\theta_{\mu\nu}u^\mu u^\nu. \tag{17}$$

The required Christoffel symbols are

$$\Gamma^\theta_{\mu\nu} = \frac{1}{2}g^{\theta\theta}\left(\partial_\mu g_{\nu\theta} + \partial_\nu g_{\mu\theta} - \partial_\theta g_{\mu\nu}\right) = \frac{2}{r}\delta_{r(\mu}\delta_{\nu)\theta} - \sin\theta\cos\theta\delta_{\mu\varphi}\delta_{\nu\varphi}. \tag{18}$$

Therefore, we have

$$\frac{d^2\theta}{d\tau^2} = -\frac{2}{r}u^r\frac{d\theta}{d\tau} + \sin\theta\cos\theta u^\varphi u^\varphi. \tag{19}$$

Starting from $\theta = \pi/2$ and $d\theta/d\tau = 0$, we see that these conditions remain satisfied at all times.

More generally, any orbit is **planar** (there is nothing special about $\theta = \pi/2$: we can change the θ, φ coordinates by a rotation). Given an orbit, we simply rotate the coordinate system to have it in the equatorial plane $\theta = \pi/2$, as this simplifies calculations.

Instead of writing the full geodesic equation, let us use the fact that ∂_t and ∂_φ are **Killing vector fields**, because the metric coefficients do not depend on t and φ. As a consequence, the covariant components u_t **and** u_φ **are constants along geodesics**. Let us give them a name:

$$E \equiv -u_t, \qquad L \equiv u_\varphi. \tag{20}$$

Physically, these play the role of energy per unit mass and angular momentum per unit mass, as can be seen from computing their values at $r \gg 2M$.

Let us now write the normalization condition for the 4-velocity (recall that $u^\theta = 0$):

$$-1 = g^{\mu\nu} u_\mu u_\nu = g^{tt}(u_t)^2 + g_{rr}(u^r)^2 + g^{\varphi\varphi}(u_\varphi)^2, \tag{21}$$

where we took advantage of the fact that $g_{\mu\nu}$ is diagonal so $g^{rr}(u_r)^2 = g_{rr}(u^r)^2$. Now recall that $u^r = dr/d\tau$, so we find

$$-1 = -\frac{E^2}{1-2M/r} + \frac{1}{1-2M/r}\left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{r^2} \qquad (\sin^2\theta = 1). \tag{22}$$

We can rewrite this as

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r) = \frac{E^2-1}{2}, \qquad V_{\text{eff}}(r) \equiv \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right) - \frac{1}{2} = -\frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}. \tag{23}$$

This is almost identical to the Newtonian equation of kineatic energy conservation for a particle orbiting about a mass M, **except** for the additional term $-ML^2/r^3$ in the effective potential. We recover the Newtonian orbits when this extra term is small relative to the other two, i.e.

$$r \gg M, \qquad r \gg L, \qquad \text{[Newtonian limit]}. \tag{24}$$

Now take one more derivative with respect to τ, and find

$$\ddot{r} = -\frac{dV_{\text{eff}}}{dr}. \tag{25}$$

We have therefore derived the radial part of the geodesic equation just by using conservation laws arising from symmetries.

Circular orbits

Let us first consider circular orbits, i.e. orbits which have a constant r. These orbits must satisfy $dV_{\text{eff}}/dr = 0$. Computing the derivative, we get

$$Mr^2 - L^2 r + 3ML^2 = 0. \tag{26}$$

The discriminant is $\Delta = L^4 - 12M^2L^2 = L^2(L^2 - 12M^2)$, which is positive only for $L > \sqrt{12}M = 2\sqrt{3}M$. We therefore conclude that there exist circular orbit only for $L > 2\sqrt{3}M$. Their radii are,

$$r_c^\pm = \frac{L^2}{2M}\left(1 \pm \sqrt{1 - 12M^2/L^2}\right). \tag{27}$$

In contrast, Newtonian circular orbits have $r_c = L^2/M$. Therefore we recover the Newtonian limit with r_c^+ when $L \gg M$.