General Relativity Fall 2018
Lecture 22: Timelike geodesics of Schwarzschild

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General equations

We have already derived the following equations by using the conservation of $E \equiv -u_t$ and $L \equiv u_\varphi$, following from the stationarity and spherical symmetry of spacetime:

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3} = \mathcal{E} \equiv \frac{E^2-1}{2}, \tag{1}$$

$$r^2\frac{d\varphi}{d\tau} = L. \tag{2}$$

Dividing the radial equation by $\dot{\varphi}^2$, we obtain the following equation for $r(\varphi)$, which entriely specifies the geometry of the orbit, even though it does not provide timescales:

$$\frac{1}{2}\left(\frac{dr}{d\varphi}\right)^2 - \frac{Mr^3}{L^2} + \frac{1}{2}r^2 - Mr = \frac{\mathcal{E}r^4}{L^2}. \tag{3}$$

Matching to Newtonian equations for Kepler problem + perturbation

Let us consider, in Newtonian theory, a test particle orbiting a central mass M, with an additional acceleration (or force per unit mass)

$$\delta\vec{f} = -3\frac{ML^2}{r^4}\hat{r}. \tag{4}$$

Such a force is the gradient of the stationary, spherically-symmetric potential

$$\delta V(r) = -\frac{ML^2}{r^3}, \tag{5}$$

so the (Newtonian) specific angular momentum vector $\vec{L} = \vec{r}\times\vec{v}$ is conserved, and so is the (Newtonian) specific energy

$$\mathcal{E} \equiv \frac{1}{2}v^2 - \frac{M}{r} - \frac{ML^2}{r^3} = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{L^2}{r^2} - \frac{M}{r} - \frac{ML^2}{r^3}, \tag{6}$$

where we used $L = r^2 d\varphi/dt$. Dividing this Newtonian energy-conservation-equation by $(d\varphi/dt)^2$, we arrive at Eq. (3) for the equation satisfied by $r(\varphi)$.

Therefore, the *geometry* of orbits $r(\varphi)$ are identical to the Newtonian problem of particles orbiting a central mass, with an additional perturbed acceleration (4). The only difference is on how time is parametrized: it is the proper time that appears in the relativistic equation, instead of the Newtonian, "universal" coordinate time.

Circular orbits and the ISCO

We rewrite Eq. (1) as

$$\frac{1}{2}\dot{r}^2 + V_{\rm eff}(r) = \mathcal{E}, \tag{7}$$

$$V_{\rm eff}(r) \equiv -\frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}. \tag{8}$$

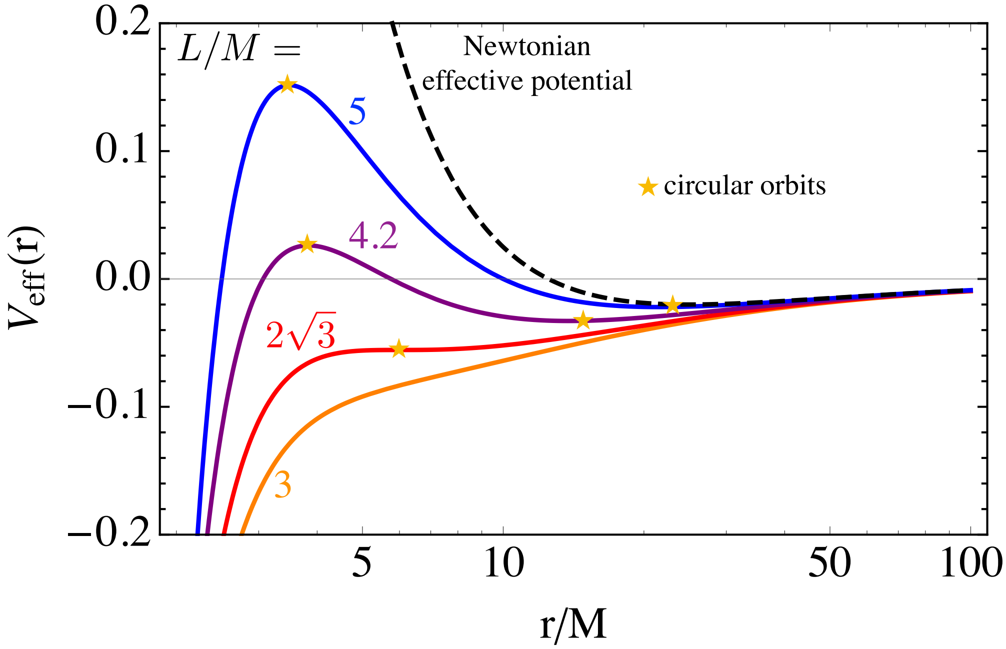

FIG. 1. Effective potential for orbits in the Schwarzschild metric. The numbers label the value of L/M. For $L/M < 2\sqrt{3}$, the potential has no extremum, and there are no circular orbits – in fact, all orbits are plunging to $r \to 2M$. For $L/M > 2\sqrt{3}$, the effective potential has a local maximum, corresponding to an unstable circular orbit, and a local minimum, corresponding to a stable circular orbit. The two merge together at $L = 2\sqrt{3}M$, for which the marginally stable circular orbit is at $r = 6M$. This is the innermost stable circular obit (ISCO).

We show the effective potential in Fig. 1. In contrast to the Newtonian effective potential for orbits around a central mass (i.e. $V_{\rm eff} \equiv -M/r + L^2/2r^2$, without the last term ML^2/r^3), which always has a minimum, the relativistic effective potential has both a maximum and a minimun for $L > 2\sqrt{3}M$, an inflection point for $L = 2\sqrt{3}M$, and is strictly monotonic for $L < 2\sqrt{3}M$.

Differentiating Eq. (7) and dividing by $dr/d\tau$, we get

$$\frac{d^2r}{d\tau^2} = -V'_{\rm eff}(r). \tag{9}$$

Cicular orbits (i.e. orbits such that r is constant) are therefore such that $V'_{\rm eff}(r) = 0$. Solving this equation, one finds that such orbits exist only for $L > 2\sqrt{3}M$. When this condition is satisfied, the radii of circular orbits are

$$r_c^{\pm} = \frac{L^2}{2M}\left(1 \pm \sqrt{1 - 12M^2/L^2}\right). \tag{10}$$

The Newtonian limit is obtained for $L \gg M$, in which case $r_c = L^2/M$.

From classical mechanics, we know that circular orbits where $d^2V_{\rm eff}/dr^2 > 0$ are stable, whereas those with $d^2V_{\rm eff}/dr^2 < 0$ are unstable. This can simply be understood graphically: for a given energy, an orbit is confined to the interval in radii for which $V_{\rm eff}(r) \leq \mathcal{E}$. At a circular orbit, this is an exact equality. Perturb the orbit away by giving it a bit of extra energy, and it will start oscillating between two radii if $d^2V_{\rm eff}/dr^2 > 0$. If $d^2V_{\rm eff}/dr^2 < 0$, it will either plunge towards $r \to 2M$ or diverge to infinity.

The smallest radius of a stable circular orbit is obtained for $L = 2\sqrt{3}M$, and has value $r_c = 6M$ – strictly speaking, only orbits with $r_c > 6M$ are actually stable. This orbit is called the **innermost stable circular orbit (ISCO)**. This is most relevant for the study of **accretion disks**: suppose material is orbiting around a black hole on a circular orbit, and slowly looses energy and angular momentum due to dissipative forces. It can remain on a quasi-circular orbit only for $r > r_{\rm isco}$, after which it will plunge towards the black hole. The specific energy at the ISCO is $\mathcal{E}_{\rm isco} = -\frac{1}{18} \approx -0.056$. Thus a particle that is brought adiabatically from infinity (where $\mathcal{E} = 0$) to the ISCO, will lose about 6% of its rest-mass energy, before it plunges towards $r = 2M$.

Precession of pericenter

We define the following variables:

$$u \equiv \frac{L^2}{Mr}, \qquad 1 - e^2 \equiv \frac{2L^2}{M^2}\mathcal{E}. \tag{11}$$

We then get

$$\left(\frac{du}{d\varphi}\right)^2 + 1 - e^2 - 2u + u^2 - 2\frac{M^2}{L^2}u^3 = 0. \tag{12}$$

Differentiating, we get

$$\frac{d^2u}{d\varphi^2} + u - 1 = 3\frac{M^2}{L^2}u^2. \tag{13}$$

In the Newtonian limit $M/L \to 0$, the right-hand-side vanishes, so $u(\varphi)$ is just a sinusoidal function: $u(\varphi) = A\cos\varphi + B\sin\varphi$. Inserting back into Eq. (12), on finds the solution $u(\varphi) = 1 + e\cos\varphi$, which implies the usual elliptical solution:

$$r(\varphi) = \frac{L^2/M}{1 + e\cos\varphi}, \qquad [\text{Newtonian limit}: \quad M/L \ll 1]. \tag{14}$$

So, in the limit $M/L \to 0$, orbits are closed ellipses, which means that the angle swept between two pericenter passages is 2π. For finite, non-zero M/L, the angle swept between two passages at minimum separation is not necessarily 2π. Instead, orbits fail to close, by an angle

$$\Delta\varphi = 2\int_{u_{\rm min}}^{u_{\rm max}} \frac{du}{du/d\varphi} - 2\pi = 2\int_{u_{\rm min}}^{u_{\rm max}} du \left(2u - u^2 - 1 + e^2 + 2\frac{M^2}{L^2}u^3\right)^{-1/2} - 2\pi, \tag{15}$$

where $u_{\rm min}$ and $u_{\rm max}$ are the roots of the cubic polynomial inside the parenthesis. Instead of computing this integral, we provide an alternate derivation, that can be used in more general context, using Gaussian perturbation equations.

Gaussian perturbation equations around Newtonian orbit

To derive the precession of pericenter, let us, for a few moments, forget all we know about GR, and just do flat-spacetime Newtonian theory. Consider a test-particle orbiting a central mass M. The following quantities are constants of motion:

$$\mathcal{E} \equiv \frac{1}{2}v^2 - \frac{M}{r} \equiv -\frac{M}{2a}, \qquad \vec{L} \equiv \vec{r} \times \vec{v}, \qquad \vec{e} \equiv \frac{1}{M}\vec{v} \times \vec{L} - \hat{r}. \tag{16}$$

The latter is the **eccentricity vector**. It is orthogonal to the specific (i.e., per-unit-mass) angular momentum $\vec{L}$. Using cylindrical coordinates (r, φ, z), we have, $\vec{r} = r\hat{r}$, $\vec{v} = \dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi}$, $\vec{L} = r^2\dot{\varphi}\hat{z} \equiv L\hat{z}$. This implies

$$\vec{v} \times \vec{L} = L\left(-\dot{r}\hat{\varphi} + r\dot{\varphi}\hat{r}\right) = -L\dot{r}\hat{\varphi} + \frac{L^2}{r}\hat{r}, \tag{17}$$

where a is the semi-major axis. This allows us to compute the norm of the eccentricity vector:

$$e^2 = \frac{1}{M^2}v^2L^2 + 1 - \frac{2}{M}\hat{r}\cdot(\vec{v} \times \vec{L}) = \frac{1}{M^2}v^2L^2 + 1 - \frac{2L^2}{Mr} = 1 + \frac{2L^2}{M^2}\mathcal{E} = 1 - \frac{L^2}{Ma}. \tag{18}$$

This implies that

$$L^2 = Ma(1 - e^2). \tag{19}$$

Therefore, given $\vec{L}$ and $\mathcal{E}$ (or a), only the orientation of the eccentricity vector really contains additional information. As can be seen from evaluating $\vec{e}$ at pericenter or apocenter, where $\dot{r} = 0$, we find that $\vec{e}$ points along the principal axis of the ellipse (it turns out it points towards pericenter).

Now suppose that, in addition to the central mass, the particle has a perturbed acceleration:

$$\ddot{\vec{r}} = -\frac{M}{r^2}\hat{r} + \delta\vec{f}. \tag{20}$$

One can still define $\mathcal{E}$, $\vec{L}$ and $\vec{e}$ at any point of the orbit (they only depend on $\vec{r}$ and $\vec{v}$), but we no longer expect them to be constants. Instead, they are now **osculating constants of motion**, i.e. the instantaneous constant of motion that the orbit would have at any given point if the perturbed force was suddenly switched off.

Consider a constant of the Kepler problem $C(\vec{r},\vec{v})$. Along the orbit, it changes with rate

$$\frac{dC}{dt} = \vec{v}\cdot\partial_{\vec{r}}C + \left(-\frac{M}{r^2}\hat{r} + \delta\vec{f}\right)\cdot\partial_{\vec{v}}C = \delta\vec{f}\cdot\partial_{\vec{v}}C, \tag{21}$$

since the first two terms cancel exactly, if C is to be an exact constant of the Kepler problem. These are called the **Gaussian perturbation equations**.

Application: radial perturbation

Let us now specialize to a perturbed acceleration of the form

$$\delta\vec{f} = -\delta V'(r)\hat{r}, \tag{22}$$

i.e. a perturbed acceleration that depends on a spherically-symmetric potential. In that case, the angular momentum remains exactly constant, and the energy needs to be supplemented by $\delta V(r)$, so that $\mathcal{E}' \equiv \frac{1}{2}v^2 - M/r + \delta V(r)$ is exactly constant. The eccentricity vector, however, is no longer constant. It is still orthogonal to $\vec{L}$, and is still aligned with $\hat{r}$ at the maximal and minimal radial distances, so it indicates the alignment of the semi-major axis. For a sufficiently small perturbation δV, the orbits are nearly Keplerian, and the eccentricity vector slowly precesses.

Expanding the double-cross product, we have

$$e^i = \frac{1}{M}\left(v^2 r^i - (\vec{v}\cdot\vec{r})v^i\right) - \hat{r}^i, \tag{23}$$

so that

$$\begin{aligned} \frac{de^i}{dt} &= -\delta V'(r)\hat{r}^j\frac{1}{M}\left(2v^j r^i - v^i r^j - (\vec{v}\cdot\vec{r})\delta^{ij}\right), \\ \Rightarrow\quad \frac{d\vec{e}}{dt} &= -\frac{1}{M}\delta V'(r)\left((\hat{r}\cdot\vec{v})\vec{r} - r\vec{v}\right) = -\frac{1}{M}\delta V'(r)(\hat{r}\times\vec{L}). \end{aligned} \tag{24}$$

The osculating eccentricity vector wobbles during each orbit. In addition, it has **secular** changes, i.e. changes on timescales much longer than the orbital timescale. To find these, average Eq. (24) over an orbital timescale T:

$$\left\langle\frac{d\vec{e}}{dt}\right\rangle = \frac{1}{M}\vec{L}\times\frac{1}{T}\int_0^T dt\delta V'(r)\hat{r} = \frac{1}{M}\vec{L}\times\frac{\int_0^{2\pi} d\varphi\ r^2\delta V'(r)\hat{r}}{\int_0^{2\pi} d\varphi\ r^2}, \tag{25}$$

where we used $\dot{\varphi} = L/r^2$. Now, on an unperturbed, elliptical orbit, we have

$$\hat{r} = \cos\varphi\ \hat{e} + \sin\varphi\ (\vec{L}\times\vec{e}), \tag{26}$$

$$r = \frac{L^2/M}{1 + e\cos\varphi}. \tag{27}$$

So we are left with

$$\left\langle\frac{d\vec{e}}{dt}\right\rangle = \Omega\hat{L}\times\vec{e}, \qquad \Omega \equiv \frac{L}{M}\frac{\int_0^{2\pi} d\varphi\ \cos\varphi\ \delta V'(r)(1+e\cos\varphi)^{-2}}{e\int_0^{2\pi} d\varphi(1+e\cos\varphi)^{-2}}. \tag{28}$$

Application to orbits around Schwarzschild

Let us now apply this to orbits around Schwarzschild, with $\delta V'(r) = 3ML^2/r^4$. We then find

$$\Omega = 3\frac{M^4}{L^5}\frac{\int_0^{2\pi} d\varphi \ \cos\varphi \ (1+e\cos\varphi)^2}{e\int_0^{2\pi} d\varphi (1+e\cos\varphi)^{-2}} = 3\frac{M^4}{L^5}(1-e^2)^{3/2} = \frac{M^{3/2}}{a^{5/2}(1-e^2)}. \tag{29}$$

The orbital time is

$$T = \int_0^T dt = \frac{1}{L}\int_0^{2\pi} d\varphi \ r^2 = 2\pi\frac{L^3}{M^2}/(1-e^2)^{3/2} = 2\pi\frac{a^{3/2}}{M^{1/2}}, \tag{30}$$

so we find that the orbit precesses by the following angle each period:

$$\Delta\varphi = \Omega T = 6\pi\frac{M^2}{L^2} = \frac{6\pi M}{a(1-e^2)}. \tag{31}$$

This is formally a correction of relative order $\mathcal{O}(v^2)$ in velocity, as it arises from the term $ML^2/r^3 \sim M/r \times L^2/r^2 \sim v^2 \times v^2$ in the effective potential.

Application to Mercury's orbit

We now apply this to the orbit of Mercury around the Sun, i.e. $M = M_\odot$, $a_{\rm M} \approx 6 \times 10^{12}$ cm, $e \approx 0.2$. This leads to a precession rate of $\Delta\varphi \approx 0.1$ arcsec per orbital period, hence $\Omega \approx 43$ arcsec/century.

It is interesting to compare this to the effect of Jupiter on the orbit. The tidal acceleration due to Jupiter is of order $M_{\rm J} a_{\rm M}/a_{\rm J}^3$. The amplitude of this perturbed acceleration relative to the relativistic correction is therefore of order

$$\frac{\delta f_{\rm Jupiter}}{\delta f_{\rm GR}} \sim \frac{M_{\rm J} a_{\rm M}/a_{\rm J}^3}{M_\odot^2/a_{\rm M}^3} \sim \frac{a_{\rm M}}{M_\odot}\frac{M_{\rm J}}{M_\odot}\left(\frac{a_{\rm M}}{a_{\rm J}}\right)^3 \sim 10^{-6}\frac{a_{\rm M}}{M_\odot} \sim 10, \tag{32}$$

where we used $M_{\rm J} \sim 10^{-3} M_\odot$ and $a_{\rm J} \sim 7 \times 10^{13}$ cm $\sim 10 \ a_{\rm M}$. Therefore we see that the relativistic precession is a factor of ~ 10 times weaker than the precession due to Jupiter's tidal force. To be aware of this tiny "anomalous precession" in the early 20-th century, physicisists therefore had to model the effect of Jupiter (and Solar-sytem planets) to exquisite precision!