General Relativity Fall 2018
Lecture 23: Schwarzschild Black holes

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Real vs coordinate singularities

We recall that the Schwarzschild metric is given by

$$ds^2 = -(1-2M/r)dt^2 + (1-2M/r)^{-1}dr^2 + r^2 d\Omega^2. \tag{1}$$

So far we have only considered orbits at $r > 2M$. The metric component g_{rr} diverges at $r \to 2M$, and so does g^{tt}. The coefficients $g^{\theta\theta}$ and $g^{\varphi\varphi}$ also diverge at $r \to 0$. Does this mean that spacetime becomes ill-behaved at these locii?

To see that it is not necessarily the case, consider flat spacetime in spherical polar coordinates, $ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$. The inverse-metric coefficient $g^{\theta\theta} = 1/r^2$ diverges at $r \to 0$, yet, we know that nothing special happens at $r = 0$: this artificial divergences just comes from our choice of coordinates.

A **sufficient** (though not necessary) condition to have a real singularity is that any of the **curvature scalars** diverge. These are scalars constructed only out of the Riemann tensor and contractions, e.g. $R, R^{\mu\nu}R_{\mu\nu}, R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}$.

For the Schwarzschild metric, $R^{\mu\nu} = R = 0$, as this is a vacuum solution. The first non-trivial curvature scalar is therefore

$$R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma} = 48\frac{M^2}{r^6}. \tag{2}$$

This diverges at $r \to 0$, which indicates that this is a real singularity. It is well-behaved at $r \to 2M$, which is not enough to conclude anything about this region.

Radially-infalling observer

To build more intuition into what happens at $r \to 2M$, consider a massive particle that is radially infalling, so that $L = 0$. Assuming infalling orbits ($dr/d\tau < 0$), for $L = 0$ the radial geodesic equation that we derived last time reduces to

$$(dr/d\tau)^2 - 2M/r = E^2 - 1 \qquad \Rightarrow \frac{dr}{d\tau} = -\sqrt{E^2 + 2M/r - 1}. \tag{3}$$

The *proper time* elapsed between some fixed radius r_0 and $r = 2M$

$$\Delta\tau = \int_{2M}^{r_0} \frac{dr}{\sqrt{E^2 + 2M/r - 1}}. \tag{4}$$

Since $E^2 \geq 0$, this is *finite*, and, for r_0 sufficiently large, of order $\Delta\tau \sim r_0/\sqrt{E^2-1}$.

Let us now compute the *coordinate time* that has elapsed. Recall that $E = -u_t = -g_{tt}u^t = (1-2M/r)dt/d\tau$. Hence, we find

$$\frac{dr}{dt} = -(1-2M/r)\sqrt{E^2 + 2M/r - 1}$$
$$\Rightarrow \Delta t = E\int_{2M}^{r_0} \frac{dr}{(1-2M/r)\sqrt{E^2 + 2M/r - 1}}. \tag{5}$$

This diverges logarithmically at $r \to 2M$.

Hence, while it takes an infinite amount of coordinate time to arrive at $r = 2M$, it only takes a finite amount of proper time to get there.

What's more, the components of the Riemann tensor in an orthonormal basis are all finite at $r \to 2M$. For instance,

$$R_{\hat{r}\hat{t}\hat{r}\hat{r}} = -2R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = -2M/r^3. \tag{6}$$

This indicates that tidal forces remain finite at $r \to 2M$. So, seemingly, nothing dramatic happens at $r \to 2M$. We now attempt to find new coordinates that allow us to smoothly describe the transition to $r \leq 2M$.

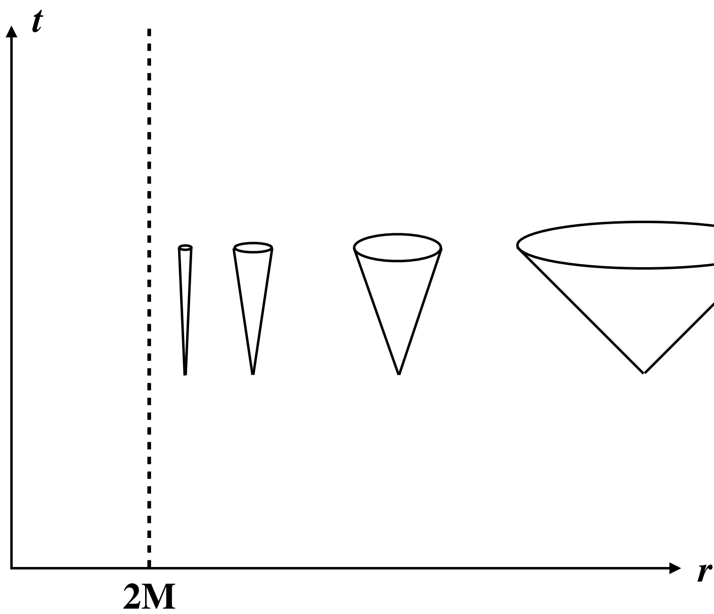

FIG. 1. Closing of light cones as $r \to 2M$ in the Schwarzschild coordinates.

Kruskal-Szekeres coordinates for the Schwarzschild black hole

We will now proceed similarly as for the Rindler spacetime $ds^2 = -x^2 dt^2 + dx^2$, in which stationary observers have a constant acceleration, and which is in fact flat spacetime in disguise (see HW 4).

We focus on the (t, r) part of the metric, hence on radial geodesics. We first study the region $r > 2M$.

The first step is to study the structure of light cones. Null radial geodesics are such that

$$\frac{dt}{dr} = \pm \left(1 - \frac{2M}{r}\right)^{-1}. \tag{7}$$

We see that the light cone "closes" as $r \to 2M$, as illustrated in Fig. 1. In these coordinates, it appears impossible to cross the region $r = 2M$. But this is just due to a poor choice of coordinates, as we will see now.

Upon integration, a photon trajectory is such that

$$t = \pm \left[r + 2M \ln(r/2M - 1)\right] + \text{constant} \equiv \pm r_* + \text{constant}. \tag{8}$$

The coordinate $r_* \in (-\infty, +\infty)$ is sometimes referred to as the "tortoise coordinate". In this coordinate, $r \to 2M$ corresponds to $r_* \to -\infty$. We now define the new variables u, v, which are constant along light rays:

$$u \equiv t - r_*, \quad du = dt - \left(1 - \frac{2M}{r}\right)^{-1} dr, \tag{9}$$

$$v \equiv t + r_*, \quad dv = dt + \left(1 - \frac{2M}{r}\right)^{-1} dr. \tag{10}$$

The inverse transformation is $t = (u + v)/2$; $r(u, v)$ is implicitly defined through

$$\frac{r}{2M} + \ln(r/2M - 1) = \frac{v - u}{4M} \quad \Rightarrow (r/2M - 1)\mathrm{e}^{r/2M} = \mathrm{e}^{\frac{v-u}{4M}}. \tag{11}$$

In these new variables, the metric takes the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du dv + r^2 d\Omega^2 = -\frac{2M}{r}\mathrm{e}^{-r/2M}\mathrm{e}^{\frac{v-u}{4M}} du dv + r^2 d\Omega^2. \tag{12}$$

Now define the variables

$$U \equiv -\mathrm{e}^{-u/4M} < 0, \quad V \equiv \mathrm{e}^{v/4M} > 0, \tag{13}$$

so that

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M} dU dV + r^2 d\Omega^2. \tag{14}$$

Finally, define

$$R \equiv \frac{V-U}{2} > 0, \quad T \equiv \frac{V+U}{2} \in (-R, +R),. \tag{15}$$

In terms of these variables, we finally arrive at

$$\boxed{ds^2 = \frac{32M^3}{r} e^{-r/2M} \left(-dT^2 + dR^2\right) + r^2 d\Omega^2}. \tag{16}$$

Let us re-express the original coordinates in terms of the new ones:

$$t = \frac{u+v}{2} = 2M\left[\ln(V) + \ln(-1/U)\right] = 2M \ln(-V/U) = 2M \ln\left(\frac{R+T}{R-T}\right), \tag{17}$$

$$\left(\frac{r}{2M} - 1\right) e^{r/2M} = R^2 - T^2. \tag{18}$$

Therefore, curves of constant t correspond to constant $T/R \in (-1, 1)$, and curves of constant $r > 2M$ are hyperbolae. This is illustrated in Fig. 2. In these coordinates, light cones are at 45 degree angles, and nothing dramatic seems to happen at $r \to 2M$.

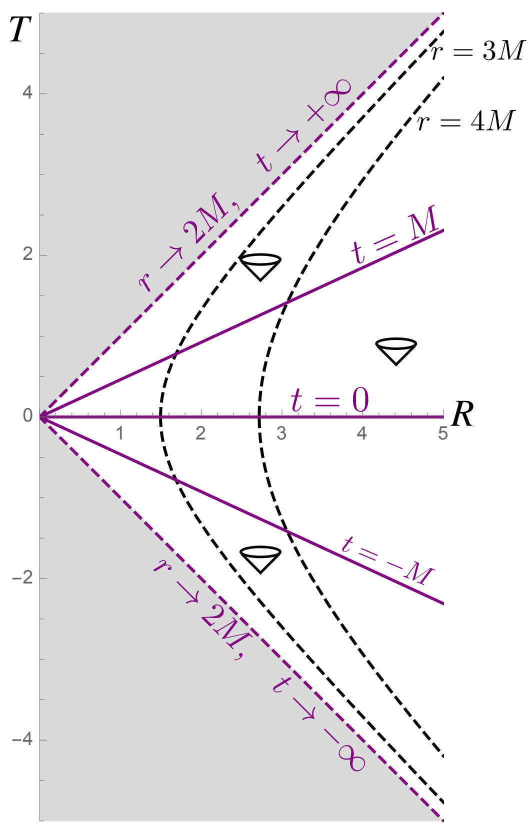

FIG. 2. Kruskal-Szekeres coordinates covering the region $t \in \mathbb{R}, r > 2M$.

Extension of the Kruskal-Szekeres coordinates

Nothing prevents us from extending theses coordinates beyond the original range for which they were constructed. In particular, the coordinate t does not appear anywhere, and we need not restrict ourselves to regions where can

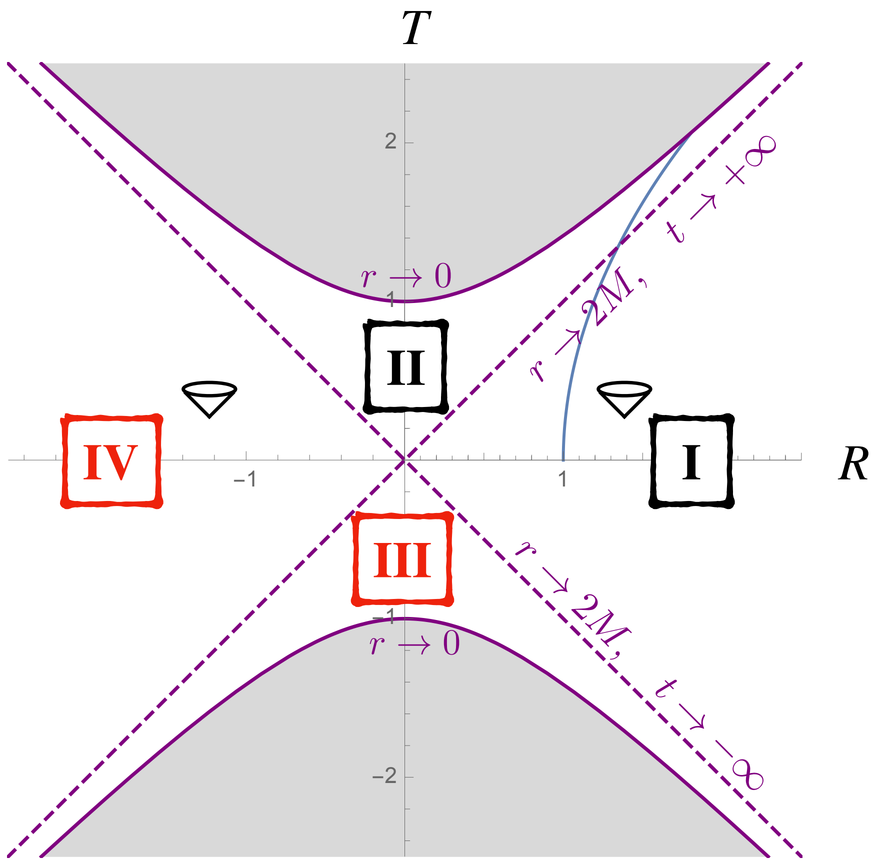

FIG. 3. Extended Kruskal-Szekeres coordinates.

explicitly define T and R in terms of t. The metric does depend on r, which is an implicit function or $R^2 - T^2$ through Eq. (18). It turns out the function $f(x) \equiv (x-1)\mathrm{e}^x$ is strictly monotonous for $x > 0$, so, for any (R, T) such that $T^2 < R^2 + 1$, there exists a unique $r > 0$. So we can extend the range of (T, R) to $R \in \mathbb{R}, |T| < \sqrt{R^2+1}$, as shown in Fig. 3. The metric is still a solution of the vacuum Einstein field equations.

The coordinate T **is always timelike, and** R **is always spacelike** (as opposed to t and r, which swicthed roles at $r < 2M$). Light cones are at 45-degree angles in these coordinates. We show an example of a timelike geodesic in blue. **Nothing special happens as it crosses the surface** $r = 2M$, and continues on.

There are 4 regions in this figure. First, region I is the $r > 2M$ region that we started with. In these coordinates, any future-directed radial timelike geodesic eventually crosses the surface[1] $r = 2M$, and, eventually, reaches the singularity $r = 0$.

The region II is what we think of the **black hole**: no future-directed trajectory in this region can ever escape it (geodesic or not!). The surface $r = 2M$ (which is a null surface) is therefore called the **horizon**. What's more, any trajectory eventually hits the **singularity** $r \to 0$, represented by $T = \sqrt{R^2+1}$, in a finite amount of proper time. Note that surfaces of constant r are spacelike (easier seen in the t, r coordinates), so the singularity $r \to 0$ is actually spacelike.

Finally, regions III and IV are "time-reversed" from I and II. Any future-directed geodesic originating from III reaches either I or IV. Region III is a "white hole". Region IV is another asymptotically flat region, that can never be reached from I. Regions III and IV are, most likely, purely mathematical, as they cannot be produced from the collapse of matter.

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[1] In fact, not all radial geodesics hit $r = 2M$, as is clear in the r, t coordinates: if a particle starts with sufficiently large energy it can escape to spatial inifnity. In the R, T coordinates, the metric is singular at $r \to +\infty$, and these coordinates are poorly adapted to describe "escape" to infinity.

CONFORMAL TRANSFORMATIONS

It is useful to be able to represent an infinite spactime in a finite diagram. These are called Penrose, or conformal diagrams. To do so, let us start with conformal trasnformations.

A conformal transformation is a rescaling of the metric by a scalar function:

$$\tilde{g}_{\mu\nu} = \omega^2(x) g_{\mu\nu}, \quad \omega^2 > 0. \tag{19}$$

Such a transformation preserves the causal structure of spacetime: null, timelike and spacelike curves remain so. Angles between vectors (as defined by the metric) are also preserved.

The Christoffel symbols of the new metric are

$$\tilde{\Gamma}^\rho_{\mu\nu} = \frac{1}{2\omega^2} g^{\rho\lambda} \left(\partial_\mu(\omega^2 g_{\nu\lambda}) + \partial_\nu(\omega^2 g_{\mu\lambda}) - \partial_\lambda(\omega^2 g_{\mu\nu})\right) = \Gamma^\rho_{\mu\nu} + \frac{1}{\omega}\left(\delta^\rho_\nu \partial_\mu \omega + \delta^\rho_\mu \partial_\nu \omega - g_{\mu\nu}\partial^\rho \omega\right) \equiv \Gamma^\rho_{\mu\nu} + \Delta\Gamma^\rho_{\mu\nu}. \tag{20}$$

Suppose a curve $x^\mu(\lambda)$ is a null geodesic of $g_{\mu\nu}$, i.e. $P^\mu = dx^\mu/d\lambda$ is such that $P^\mu \nabla_\mu P^\nu = 0, P^\mu P_\mu = 0$. Let us define $\tilde{\lambda} \equiv f(\lambda)$, and $\tilde{P}^\mu \equiv dx^\mu/d\tilde{\lambda} = P^\mu/f'(\lambda)$. This vector is null, and is such that

$$\begin{aligned}\tilde{P}^\mu \tilde{\nabla}_\mu \tilde{P}^\nu &= \frac{1}{f'} P^\mu \tilde{\nabla}_\mu (P^\nu/f') = \frac{1}{f'^2} P^\mu \tilde{\nabla}_\mu P^\nu + \frac{1}{f'} P^\nu P^\mu \tilde{\nabla}_\mu (1/f') = \frac{1}{f'^2} P^\mu \nabla_\mu P^\nu + \frac{1}{f'^2} P^\mu \Delta\Gamma^\nu_{\mu\sigma} P^\sigma - \frac{1}{f'^3} P^\nu P^\mu \partial_\mu (f') \\ &= 0 + \frac{1}{f'^2}\frac{1}{\omega}\left(2P^\nu P^\mu \partial_\mu \ln\omega\right) - \frac{1}{f'^3} P^\nu P^\mu \partial_\mu(f') = \frac{P^\nu}{f'^2}\frac{d}{d\lambda}(\omega^2/f'). \end{aligned} \tag{21}$$

Hence we see that if we choose $d\tilde{\lambda}/d\lambda = \omega^2$ (or a constant times that), the curves $x^\mu(\tilde{\lambda})$ are null geodesics of $\tilde{g}_{\mu\nu}$. So a **conformal transformation preserves null geodesics**, though it does not preserve timelike geodesics.

Finally, the Weyl tensor $W^\rho_{\mu\nu\lambda}$ (the fully traceless part of Riemann) is identical in the two metrics – specifically with this placement of indices.